Definitions for Scheme Theory

Robert L. Bocchino Jr.

Revised February 2024

This document presents basic definitions associated with the theory of **schemes**. A scheme is an algebraic and geometric construct whose definition is motivated by the concept of an algebraic variety in classical algebraic geometry. All of the classical geometry of varieties can be restated in the language of schemes. Doing this simplifies the proofs of many classical results. Further, scheme theory enables new algebraic geometry that is not possible with the classical methods. Therefore, schemes have replaced varieties as the fundamental objects of study in the field of algebraic geometry.

This paper assumes that you are familiar with the material covered in my papers *Definitions for Commutative Algebra* and *Definitions for Classical Algebraic Geometry*. Throughout this document, when we say "ring" we shall mean "commutative ring."[1]

One of the hallmarks of the scheme-theoretic approach to algebraic geometry is a deep connection to category theory. In this paper, I have tried to minimize the dependence on the technical details of category theory. You should know the definition of a category, which is given in § 2 of my paper *Definitions for Category Theory*. Up to § 7 of this paper you don't need to know more than that; the connections to category theory are discussed in footnotes, with citations pointing out where you can learn more. Section 7 requires knowledge of some technical concepts from category theory. If those concepts are unfamiliar, you can read the material cited there, or you can skip that section.

1. Affine Varieties

We begin by restating the definition of an affine variety in a way that will naturally lead to the definition of an affine scheme (§ 4). As in *Definitions for Classical Algebraic Geometry*, let K be an algebraically closed field, let n be a natural number greater than zero, and let $\mathbf{A}^n$ be the n-dimensional affine space over K. A point a of $\mathbf{A}^n$ is an n-tuple of coordinates: $a = (a_1, \ldots, a_n) = \{a_i\}$. Let $K[z]$ denote the polynomial ring $K[z_1, \ldots, z_n]$, and let $p(z)$ denote a polynomial $p(z_1, \ldots, z_n)$ in $K[z]$.

Recall that an affine variety $V \subseteq \mathbf{A}^n$ is the zero set $V(F)$ of a family of polynomials $F = \{p_\alpha\}$ in $K[z]$. That is, V is the set of points a in $\mathbf{A}^n$ such that $p_\alpha(a) = 0$ for all p_α in F. The recipe for computing $p_\alpha(a)$ is the usual one, i.e., plug in the coordinate a_i for each variable z_i and collect terms. Note that this recipe relies on the representation of the point a in terms of its coordinates $\{a_i\}$. We wish to restate the definition of V in a way that does not rely on this representation.

First, recall that there is a one-to-one correspondence between the points of $\mathbf{A}^n$ and the maximal ideals of the ring $K[z]$; this correspondence associates to each point $a = \{a_i\}$ in $\mathbf{A}^n$ the maximal ideal generated by the polynomials $\{z_i - a_i\}$. This ideal, which we denote $\mathbf{m}_a$, contains exactly the polynomials in $K[z]$ that vanish at a. See *Definitions for Classical Algebraic Geometry*, § 7.1. Thus we can move from considering points a in $\mathbf{A}^n$ to considering maximal ideals $\mathbf{m}$ of $K[z]$, and this representation is coordinate-free.

Next, observe that for any maximal ideal $\mathbf{m}$ of $K[z]$, the residue field $K[z]/\mathbf{m}$ of $K[z]$ modulo $\mathbf{m}$ is isomorphic to K. See *Definitions for Commutative Algebra* § 6. Further, if p is a polynomial in $K[z]$ and a is a point of $\mathbf{A}^n$, then we can write

$$p(z) = q(z) + p(a),$$

where $q(z) = p(z) - p(a)$. Since $q(a) = 0$, q is an element of $\mathbf{m}_a$. Therefore the coset containing $p(z)$ in $K[z]/\mathbf{m}_a$ is $\mathbf{m}_a + p(a)$, which corresponds to the element $p(a)$ of K. Thus we have a recipe for evaluating polynomials in $K[z]$

[1] Some authors say "commutative ring with identity." Here we use the definition given in § 6 of *Definitions for Commutative Algebra*, which states that the nonzero elements of a ring R form a multiplicative monoid, so R has a multiplicative identity unless it is the zero ring. This definition is standard.

without referring to coordinates: for any polynomial p and any maximal ideal $\mathbf{m}$, let $p(\mathbf{m})$ be the element of K corresponding to the coset of $K[z]/\mathbf{m}$ of which p is a representative.

Now we have a coordinate-free way to define an affine variety. Fix a family $F = \{p_\alpha\}$ of polynomials in $K[z]$. Define the affine variety $V(F)$ to be the set of all maximal ideals $\mathbf{m}$ of $K[z]$ such that $p_\alpha(\mathbf{m}) = 0$ for all p_α in F.

We can use the same method to define the Zariski topology on $\mathbf{A}^n$. Recall that the closed sets in the Zariski topology are the affine varieties $V \subseteq \mathbf{A}^n$. See *Definitions for Classical Algebraic Geometry*, § 5.1. Thus the closed sets are exactly the sets $V(F)$, where F is a family of polynomials in $K[z]$.

Note the following:

1. For each α, p_α corresponds to the zero coset of $K[z]/\mathbf{m}$ if and only if p_α is a member of $\mathbf{m}$, considered as a set of polynomials.
2. Therefore $V(F)$ is the set of maximal ideals $\mathbf{m}$ of $K[z]$ such that p_α is a member of $\mathbf{m}$ for all α.

Thus to formulate the concept of an algebraic variety, we don't need to evaluate $p_\alpha(\mathbf{m})$ at all; we can just ask whether p_α is a member of $\mathbf{m}$. This move from the question "Does p evaluate to zero at a point?" to the question "Is p a member of a particular ideal?" is fundamental to scheme theory.

If we do think about evaluating functions p at points $\mathbf{m}$, then we just have to accept the weirdness that the functions we evaluate are members of the points where we evaluate them. This is not how functions and points normally work in mathematics. One way to think about this is as follows: we represent a point $\mathbf{m}$ as a set of functions, i.e., all and only the functions p that evaluate to zero at $\mathbf{m}$.

2. Sheaves

In this section we define the concept of a **sheaf**, which is a fundamental construction in scheme theory. A sheaf is a topological space T together with a specification, for each open set U of T, of some local data associated with U. The local data must satisfy several compatibility conditions on the areas of overlap between the open sets of T.

As with many ideas in modern mathematics, the concept of a sheaf is very abstract and general but is motivated by examples that occur in practice. One can think of a sheaf as a "pattern" that appears in various applications in differential geometry, algebraic geometry, etc.

2.1. The Sheaf of Regular Functions on an Affine Variety

We begin with an example from classical algebraic geometry: the sheaf of regular functions on an affine variety.

Definition: Let $V \subseteq \mathbf{A}^n$ be an affine variety, and let $U \subseteq V$ be an open subset of V in the Zariski topology on V (§ 1). Let $f: U \to K$ be a function, and let a be a point of U. Recall that f is **regular at** a if there exist a set $W \subseteq U$ open in U containing a and polynomials p and q in $K[z]$ such that, for all b in W, (a) $q(b) \neq 0$ and (b) $f(b) = p(b)/q(b)$. Recall that f is **regular on** U if it is regular at every point a in U. See *Definitions for Classical Algebraic Geometry*, § 7.1.

For each open set U in V, let $\boldsymbol{O}(U)$ denote the set of regular functions on U. $\boldsymbol{O}(U)$ is a ring, under the standard rules for adding and multiplying polynomial fractions. See *Definitions for Classical Algebraic Geometry*, § 7.1.

Let U and W be open subsets of V, with $U \subseteq W$. Define the **restriction map** $\boldsymbol{O}_{W,U}: \boldsymbol{O}(W) \to \boldsymbol{O}(U)$ as follows:

$$\boldsymbol{O}_{W,U}(f) = f|_U.$$

That is, the image of f under $\boldsymbol{O}_{W,U}$ is f restricted to U.

Together, the rings $\boldsymbol{O}(U)$ and the maps $\boldsymbol{O}_{W,U}$ are called the **sheaf of regular functions** on V. We will denote this sheaf $\boldsymbol{O}$.

Properties: Let O denote the set of open subsets of V. Observe that $\boldsymbol{O}$ has the following properties:

S1. For any set U in O, $\boldsymbol{O}_{U,U}$ is the identity map.

S2. For any sets $U \subseteq W \subseteq X$ in O, we have $\boldsymbol{O}_{W,U} \circ \boldsymbol{O}_{X,W} = \boldsymbol{O}_{X,U}$.

S3. Let U be set in O, and let S be a set of sets in O such that $\cup S = U$. For each set X in S, fix an element f_X in $\boldsymbol{O}(X)$. Suppose that for each pair (X, Y) of sets in S we have

$$\boldsymbol{O}_{X,X \cap Y}(f_X) = \boldsymbol{O}_{Y,X \cap Y}(f_Y).$$

Then there is a unique element f in $\boldsymbol{O}(U)$ such that $\boldsymbol{O}_{U,X}(f) = f_X$ for all X in S.

As to property **S3**, for any a in U, we can set $f(a)$ to be $f_X(a)$, for any X in S that contains a; the conditions ensure that (a) some such X exists and (b) the definition of f is independent of the choice of X.

The stalk of the sheaf at a point: Let a be a point of V. Let S be the set of all pairs (U, f), where U is open in V, U contains a, and $f: U \to K$ is a regular function. Construct a ring G from S by identifying pairs of elements that satisfy the following equivalence relation: $(U, f) \sim (W, g)$ if there exists an open set $X \subseteq U \cap W$ such that $a \in X$, and f and g agree on X. Recall that the elements of G are called **regular function germs**, because they characterize the local behavior of a class of functions, each of which is defined in a neighborhood of a.

In the terminology of sheaf theory, the ring G is called the **stalk** of the sheaf $\boldsymbol{O}$ at the point a and written $\boldsymbol{O}_a$. The elements of $\boldsymbol{O}_a$ are called the **germs** of the stalk.

Recall that if V is irreducible, then G is exactly the local ring of V at a, which we also denoted $\boldsymbol{O}_a(V)$. See *Definitions for Classical Algebraic Geometry*, § 7.1.

The stalk as a direct limit: We may express the stalk $\boldsymbol{O}_a$ as a direct limit as follows. Let $O_a \subseteq O$ be the set of all open sets of V that contain a. Give O_a the following partial order: $U \le W$ if $W \subseteq U$ (note the reverse inclusion). Then O_a is a directed set, i.e., a nonempty partially ordered set in which every pair of elements has an upper bound. Further, the restriction maps $\boldsymbol{O}_{U,W}$ make the family $\{\boldsymbol{O}(U)\}_{U \in O_a}$ into a direct system of modules over O_a (remember that a ring R is an R-module). Comparing the definition of the stalk $\boldsymbol{O}_a$ to the definition of the direct limit, we see that

$$\boldsymbol{O}_a = \varinjlim \{\boldsymbol{O}(U)\}_{U \in O_a}.$$

See *Definitions for Commutative Algebra*, § 10.

2.2. General Sheaves

We now give the general definition of a sheaf. Let $T = (S, O)$ be a topological space with underlying set S and open sets O. Let C be the pair (M, μ), where (1) M is a set of sets, possibly with additional structure; and (2) μ is a set of mappings between the elements of M that satisfy the usual rules for identity maps and composition of maps. For example, M could be a collection of sets, and μ could be set mappings; M could be a collection of groups, rings, or modules, and μ could be homomorphisms; etc.[2]

Presheaves: First we define the concept of a presheaf. A presheaf is similar to a sheaf, except that it may be missing some local data. Presheaves are useful because they generalize sheaves, and because we can complete them to sheaves in a canonical way (§ 2.4).

Let P be the set of all pairs of sets (X, Y) in $O \times O$ such that $Y \subseteq X$. A **presheaf $\boldsymbol{F}$** from T to C is a pair of maps:

1. A map $\boldsymbol{F}: O \to M$ that assigns, to each open set U in O, an object $\boldsymbol{F}(U)$ in M.
2. A map $\boldsymbol{F}: P \to \mu$ that assigns, to each pair (X, Y) in P, a map $\boldsymbol{F}(X, Y): \boldsymbol{F}(X) \to \boldsymbol{F}(Y)$.

We write $\boldsymbol{F}_{X,Y}$ to denote $\boldsymbol{F}(X, Y)$.

The maps must satisfy properties **S1** and **S2** stated in the previous section, but not necessarily property **S3**. The elements of $\boldsymbol{F}(U)$ are called the **sections of $\boldsymbol{F}$ over** U, and the elements of $\boldsymbol{F}(S)$ are called **global sections**. The maps $\boldsymbol{F}_{X,Y}$ are called **restriction maps**.[3]

Recall that in the sheaf of regular functions on an affine variety (§ 2.1), the sections are functions, and each restriction map is an actual restriction, in the sense of restricting a function to a subset of its domain. For a general sheaf, these statements need not be true.

Let $\boldsymbol{F}$ and $\boldsymbol{G}$ be presheaves from T to C. We say that **$\boldsymbol{G}$ extends $\boldsymbol{F}$** if (1) $\boldsymbol{F}(U) \subseteq \boldsymbol{G}(U)$ for each $U \in O$; and (2) $\boldsymbol{F}(X, Y) \subseteq \boldsymbol{G}(X, Y)$ (treating the maps as sets of ordered pairs) for each $(X, Y) \in P$.

When C consists of sets and set mappings, we say that $\boldsymbol{F}$ is a **presheaf of sets**. When C consists of modules and module homomorphisms, we say that $\boldsymbol{F}$ is a **presheaf of modules**. Similarly, we define a presheaf of groups or a presheaf of rings.

[2] In other words, C is a concrete category. See §§ 2 and 6 of my paper *Definitions for Category Theory*.

[3] Let B be the category whose objects are the elements of O and whose arrows are the set inclusion maps. A presheaf $\boldsymbol{F}$ from T to C is a contravariant functor $\boldsymbol{F}$ from B to C. See *Definitions for Category Theory*, § 6.

Sheaves: A **sheaf** $\boldsymbol{F}$ from T to C is a presheaf $\boldsymbol{F}$ from T to C that satisfies property **S3** stated in the previous section. This property is called the **sheaf axiom**. The sheaf axiom ensures that if we have two sections f_1 defined on U_1 and f_2 defined on U_2, and f_1 and f_2 are defined consistently on the overlap between U_1 and U_2, then there is a section f on $U_1 \cup U_2$ that restricts to f_i on each U_i. In the case where the sections are maps, one can imagine "gluing" or "pasting" the function graphs on the smaller domains to construct a function graph on the larger domain. The sheaf axiom ensures that this gluing or pasting operation always yields a section of the sheaf.

For example:

1. Let $\boldsymbol{F}^\infty$ denote the sheaf of sets of infinitely differentiable real-valued functions on the open sets of $\mathbf{R}$, with the restriction maps $\boldsymbol{F}_{U,W} = f \mapsto f|_W$. It is easy to see that $\boldsymbol{F}$ is a presheaf, and that it satisfies the sheaf axiom.
2. Now let $U = (-2, 2)$, and let $f \in \boldsymbol{F}(U)$ be the function $x \mapsto 1$. Let $\boldsymbol{P}$ be the presheaf given by deleting f from $\boldsymbol{F}(U)$. $\boldsymbol{P}$ does not satisfy the sheaf axiom and so is not a sheaf. Indeed, let $X = (-2, 1)$, let $Y = (0, 2)$, let $f_X \in \boldsymbol{G}(X) = x \mapsto 1$ on X, and let $f_Y \in \boldsymbol{G}(Y) = y \mapsto 1$ on Y. Then f_X and f_Y agree on the intersection $X \cap Y = (0, 1)$, but there is no function f defined on U that restricts to f_X on X and restricts to f_Y on Y, because that function is f, which we deleted.

We define a sheaf of groups, rings, or modules similarly to a presheaf of groups, rings, or modules.

The stalk of a sheaf at a point: Let $\boldsymbol{F}$ be a sheaf from $T = (S, O)$ to C, let a be a point of S, and let $O_a \subseteq O$ be the set of open sets U such that $a \in U$. The **stalk** of $\boldsymbol{F}$ at a, written $\boldsymbol{F}_a$, is the set of all pairs (U, f) where U is a member of O_a, and f is a member of $\boldsymbol{F}(U)$, under the following equivalence relation: $(U, f) \sim (W, g)$ if there exists a set X in O_a such that $X \subseteq U \cap W$, and $\boldsymbol{F}_{U,X}(f) = \boldsymbol{F}_{W,X}(g)$. When the sets $\boldsymbol{F}(U)$ are modules, the stalk $\boldsymbol{F}_a$ is the direct limit

$$\boldsymbol{F}_a = \varinjlim \{\boldsymbol{F}(U)\}_{U \in O_a}$$

as discussed in the previous section.

We write $[U, f]_a$ to denote the equivalence class of the pair (U, f) in the stalk $\boldsymbol{F}_a$.

For example, let $\boldsymbol{F} = \boldsymbol{F}^\infty$, and let a be a point of $\mathbf{R}$. Let f and g be two infinitely differentiable functions with domains U and V that contain a. Then $(U, f) \sim (V, g)$ in the stalk $\boldsymbol{F}_a$ if and only if f and g have the same Taylor series expansion at a.[4] Therefore we may think of the stalk $\boldsymbol{F}_a$ as follows:

1. $\boldsymbol{F}_a$ is the set of all Taylor series expansions t at a of infinitely differentiable functions whose domains of definition include a.
2. An element $t \in \boldsymbol{F}_a$ is represented by any infinitely differentiable function f whose domain U includes a and whose Taylor series expansion at a is t. In this case we have $[U, f]_a = t$.

2.3. Morphisms of Sheaves

Let $C = (M, \mu)$, and let $D = (M', \mu')$. Let $\boldsymbol{F}$ be a presheaf from $T = (S, O)$ to C, and let $\boldsymbol{G}$ be a presheaf from T to D. A **morphism** $\phi\colon \boldsymbol{F} \to \boldsymbol{G}$ is a family of maps $\{\phi_U\colon \boldsymbol{F}(U) \to \boldsymbol{G}(U)\}_{U \in O}$, one for each set in O, such that for every inclusion $U \subseteq W$ of sets in O, the diagram shown in Figure 1 commutes.[5] Since every sheaf is a presheaf, the same definition applies if $\boldsymbol{F}$ or $\boldsymbol{G}$ or both are sheaves.

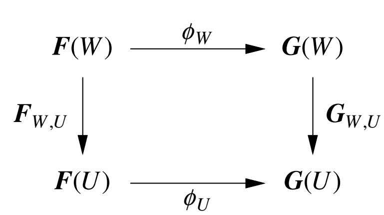

Figure 1: The commutative diagram for a morphism $\phi\colon \boldsymbol{F} \to \boldsymbol{G}$ of presheaves or sheaves.

Note that if $\boldsymbol{F}$ and $\boldsymbol{G}$ are presheaves of modules (or rings, or groups), then the maps ϕ_U are module (or ring, or

[4] For a discussion of Taylor series expansions, see § 9 of my paper *The General Derivative*.

[5] In other words, ϕ is a natural transformation from the functor $\boldsymbol{F}$ to the functor $\boldsymbol{G}$. See *Definitions for Category Theory*, § 7.

group) homomorphisms. If there exists a morphism $\psi: \boldsymbol{G} \to \boldsymbol{F}$ such that $\psi_U \circ \phi_U = \phi_U \circ \psi_U = id$ for all U, then we say that ϕ is an **isomorphism**.

For each point a in S, the morphism ϕ induces a map $\phi_a: \boldsymbol{F}_a \to \boldsymbol{G}_a$ between the stalks.

Injective and surjective morphisms: Assume that $\boldsymbol{F}$ and $\boldsymbol{G}$ are sheaves.

1. ϕ is an isomorphism if and only if each of the maps ϕ_a is an isomorphism (proof omitted).
2. If each map ϕ_a is injective, then we say that ϕ is **injective**; and similarly if each map ϕ_a is surjective or bijective. We claim the following (proofs omitted):
   a. ϕ is injective if and only if each map ϕ_U is injective.
   b. If the maps ϕ_U are surjective, then ϕ is surjective.

   Note that the converse of item 2(b) is not true.

Now assume only that $\boldsymbol{F}$ and $\boldsymbol{G}$ are presheaves.

1. We say that ϕ is **injective** if each map ϕ_U is injective.
2. We define the concept of a surjective morphism $\phi: \boldsymbol{F} \to \boldsymbol{G}$ in the next section, after we define the concept of sheafification.

Subsheaves: Let $\boldsymbol{F}$ and $\boldsymbol{G}$ be sheaves on T, and let $\phi: \boldsymbol{F} \to \boldsymbol{G}$ be an injective morphism. In this case we say that $\boldsymbol{F}$ is a **subsheaf** of $\boldsymbol{G}$.

2.4. Sheafification

In this section we describe a process called **sheafification** for extending a presheaf to a sheaf.

The presheaf of stalk maps: Let $\boldsymbol{P}$ be a presheaf from $T = (S, O)$ to C. and let D be the disjoint union $\biguplus_{a \in S} \boldsymbol{P}_a$ of the stalks $\boldsymbol{P}_a$ at the points a of U. That is,

$$D = \{(a, x): a \in S, x \in \boldsymbol{P}_a\}.$$

For each open set U in O, make the following constructions:

1. For each p in $\boldsymbol{P}(U)$, define $f_p: U \to D$ to be the map $a \mapsto (a, [U, p]_a)$. For example, if $\boldsymbol{P} = \boldsymbol{F}^\infty$, then $f_p(a) = (a, t_{p,a})$, where $t_{p,a}$ represents the Taylor series expansion of p at a.
2. Let $\boldsymbol{Q}(U)$ be the set of mappings $\{f_p\}_{p \in P(U)}$.

We claim the following (proofs omitted):

1. The sets $\boldsymbol{Q}(U)$, together with the restriction maps $\boldsymbol{Q}_{U,W} = f \mapsto f|_W$, form a presheaf of sets on T. We call $\boldsymbol{Q}$ the **presheaf of stalk maps** corresponding to the presheaf $\boldsymbol{P}$.
2. For each open set U in O, let $\phi_U: \boldsymbol{P}(U) \to \boldsymbol{Q}(U)$ be the map $p \mapsto f_p$. The family of maps $\phi = \{\phi_U\}$ is an isomorphism of presheaves. We call ϕ the **morphism to the presheaf of stalk maps**.

This construction lets us represent any presheaf $\boldsymbol{P}$ as a presheaf $\boldsymbol{Q}$ in which the sections are maps. In the case $\boldsymbol{P} = \boldsymbol{F}^\infty$, the map ϕ takes each function p to a map f_p that records, for each point a in the domain of p, the Taylor series expansion of p at a.

Sheafification: We can extend the presheaf of stalk maps to a sheaf by adding sections to it, as follows. Let $\boldsymbol{P}$, $\boldsymbol{Q}$, D, and U be as in the discussion above, and let $f: U \to D$ be a mapping such that for each a in U, $f(a)$ is an element of $\{a\} \times P_a$. Let $X = \{(U_i, p_i)\}$ be a family of pairs such that

1. The U_i are open sets of O, and $\cup U_i = U$.
2. For each i, p_i is an element of $\boldsymbol{P}(U_i)$.

We say that f is **locally induced** by the family X if, for each i, and for each point a in U_i, $f(a) = (a, [U_i, p_i]_a)$. Note that each mapping $f_p \in \boldsymbol{Q}(U)$ as defined above is locally induced by the single pair (U, p).

Let $\boldsymbol{F}(U)$ be the set of all mappings $f: U \to D$ such that f is locally induced by a family of pairs X. Note that $\boldsymbol{F}(U) \supseteq \boldsymbol{Q}(U)$. The sets $\boldsymbol{F}(U)$, together with the restriction maps $\boldsymbol{F}_{U,W} = f \mapsto f|_W$, form a sheaf from T to C that extends $\boldsymbol{Q}$ (§ 2.2) (proof omitted). We call $\boldsymbol{F}$ the **sheafification** of the presheaf $\boldsymbol{P}$.

For example, let $\boldsymbol{P}$ be the presheaf given by deleting the function $g(x) = 1$ with domain $U = (-2, 2)$ from $\boldsymbol{F}^\infty$. Then $\boldsymbol{F}(U)$ includes the function $f_g(x) = (x, t(g, x))$, even though g is missing from $\boldsymbol{P}$. This is because f_g is locally induced (for example) by the pairs $(X, g|_X)$ and $(Y, g|_Y)$, where $X = (-2, 1)$ and $Y = (0, 2)$.

The universal property of sheafification: Let $T = (S, O)$ be a topological space, let $\boldsymbol{P}$ be a presheaf from T to C, and let $\boldsymbol{F}$ be the sheafification of T. Let $\phi: \boldsymbol{P} \to \boldsymbol{F}$ be defined as for the morphism to the presheaf of stalk maps.

1. $\boldsymbol{F}$ and ϕ have the following universal property: for any presheaf $\boldsymbol{Q}$ from T to C and any morphism $\psi: \boldsymbol{P} \to \boldsymbol{Q}$, there exists a unique morphism $\psi': \boldsymbol{F} \to \boldsymbol{Q}$ such that $\psi = \psi' \circ \phi$ (proof omitted).
2. It follows easily from item 1 that if $\boldsymbol{P}$ is a sheaf, then $\boldsymbol{P}$ is isomorphic to its sheafification $\boldsymbol{F}$.
3. For each $a \in S$, the induced map $\phi_a: \boldsymbol{P}_a \to \boldsymbol{F}_a$ is an isomorphism. Further, given $\boldsymbol{P}$, $\boldsymbol{F}$ and ϕ are the unique sheaf and presheaf morphism with this property (proof omitted).

Surjective morphisms of presheaves: Let T be a topological space, let $\boldsymbol{F}$ and $\boldsymbol{G}$ be presheaves from T to C, and let $\phi: F \to G$ be a morphism. Construct the presheaf $\boldsymbol{P}$ from T to C as follows:

1. $\boldsymbol{P}(U) = \phi_U(\boldsymbol{F}(U)) \subseteq \boldsymbol{G}(U)$.
2. $\boldsymbol{P}_{X,Y}$ is $\boldsymbol{G}_{X,Y}$ restricted to $\boldsymbol{P}(U)$.

Let $\boldsymbol{H}$ be the sheafification of $\boldsymbol{P}$, and let $\psi: \boldsymbol{H} \to \boldsymbol{G}$ be the morphism induced by the inclusion morphism from $\boldsymbol{P}$ to $\boldsymbol{G}$ and by the universal property of sheafification.

1. We say that ϕ is **surjective** if $\psi_U(\boldsymbol{H}(U)) = \boldsymbol{G}(U)$ for all U.
2. If $\boldsymbol{F}$ and $\boldsymbol{G}$ are sheaves, then this definition agrees with the definition given in § 2.3 for a surjective morphism of sheaves (proof omitted).

2.5. Constructions on Sheaves

In this section we describe several useful constructions on presheaves and sheaves. We will use these constructions in later sections.

The quotient of a sheaf by a sheaf: Fix a topological space $T = (S, O)$. Let R be a ring, let $\boldsymbol{F}$ and $\boldsymbol{G}$ be sheaves of R-modules on T, and suppose there exists an injective morphism $\phi: \boldsymbol{F} \to \boldsymbol{G}$.

1. The **quotient presheaf** of $\boldsymbol{G}$ by $\boldsymbol{F}$ is the presheaf $\boldsymbol{P}$ given by the following rules:
   a. For each open set U in O, $\boldsymbol{P}(U) = \boldsymbol{G}(U)/\boldsymbol{F}(U)$.
   b. For each pair of open sets X and Y in O with $Y \subseteq X$, $\boldsymbol{P}_{X,Y}$ is the map that takes the equivalence class $[x]$ in $\boldsymbol{G}(X)/\boldsymbol{F}(X)$ to the equivalence class $[\boldsymbol{G}_{X,Y}(x)]$ in $\boldsymbol{G}(Y)/\boldsymbol{F}(Y)$.
2. The **quotient sheaf** of $\boldsymbol{G}$ by $\boldsymbol{F}$, written $\boldsymbol{G}/\boldsymbol{F}$, is the sheafification of $\boldsymbol{P}$.

The pushforward of a presheaf: Let $T_1 = (S_1, O_1)$ and $T_2 = (S_2, O_2)$ be topological spaces, and let $\phi: T_1 \to T_2$ be a continuous map. Suppose that $\boldsymbol{P}$ is a presheaf from T_1 to $C = (M, \mu)$. The **pushforward** or **direct image** of $\boldsymbol{P}$ by ϕ, written $\phi_* \boldsymbol{P}$, is the presheaf from T_2 to C given by the following rules:

1. For each open set U in O_2, $(\phi_* \boldsymbol{P})(U) = \boldsymbol{P}(\phi^{-1}(U))$.
2. For each pair of open sets X and Y in O_2 with $Y \subseteq X$, $(\phi_* \boldsymbol{P})_{X,Y} = \boldsymbol{P}_{\phi^{-1}(X), \phi^{-1}(Y)}$.

The pushforward of a sheaf is a sheaf (proof omitted).

The restriction of a presheaf: Let $\boldsymbol{P}$ be a presheaf from $T = (S, O)$ to $C = (M, \mu)$. Let $U \subseteq S$ be an open set.

1. We write $T|_U$ to denote the topological space (U, O'), where O' is induced by O via the subset topology.
2. We write $\boldsymbol{P}|_U$ to denote the presheaf from $T|_U$ to C obtained by restricting the maps of $\boldsymbol{P}$ to the domains provided by $T|_U$.

We call $\boldsymbol{P}|_U$ the **restriction** of the presheaf $\boldsymbol{P}$ to the open set U. If $\boldsymbol{P}$ is a sheaf, then $\boldsymbol{P}|_U$ is also a sheaf (proof omitted). If $\phi: \boldsymbol{P} \to \boldsymbol{P}'$ is a morphism of presheaves, then we write $\phi|_P$ to denote the induced map from $\boldsymbol{P}|_U$ to $\boldsymbol{P}'$.

***B*-sheaves:** Let $T = (S, O)$ be a topological space. Recall that a subset $B \subseteq O$ is called a **base** or **basis** for T if each open set in O may be expressed as a union of sets in B. We also say that the sets in B are **basic open sets** of T.

Let B be a base for T. Let $C = (M, \mu)$ be as in the definition of a sheaf. A ***B*-sheaf** from T to C is like a sheaf from T to C, but it is defined on the sets B. Formally, we define a B-sheaf $\boldsymbol{F}$ in the same way as a sheaf $\boldsymbol{F}$, with the following amendments:

1. We replace O with B everywhere in the definition.
2. We rewrite the sheaf axiom **S3** as follows:

   S3. Let U be set in B, and let S be a set of sets in B such that $\cup S = U$. For each set X in S, fix an element f_X in $\boldsymbol{F}(X)$. Suppose that for each pair (X, Y) of sets in S, and for each set $W \in B$ such that $W \subseteq X \cap Y$, we have

   $$\boldsymbol{F}_{X,W}(f_X) = \boldsymbol{F}_{Y,W}(f_Y).$$

   Then there is a unique element f in $\boldsymbol{F}(U)$ such that $\boldsymbol{F}_{U,X}(f) = f_X$ for all X in S.

Let $\boldsymbol{F}$ be a B-sheaf on T. Then there is a unique sheaf $\boldsymbol{G}$ on T such that $\boldsymbol{G}$ and $\boldsymbol{F}$ agree on the basic open sets of T. See [Eisenbud and Harris 1999], Proposition I-12. Therefore we may specify a sheaf by specifying a B-sheaf.

3. Ringed Spaces

In scheme theory we are primarily concerned with a special kind of topological space equipped with a sheaf, called a **ringed space**. A ringed space is a pair $(T, \boldsymbol{O})$, where T is a topological space, and $\boldsymbol{O}$ is a sheaf of commutative rings on T (§ 2.2). An affine variety together with its sheaf of regular functions (§ 2.1) is a ringed space. In later sections we shall see that schemes are also ringed spaces.

In the rest of this section, let $T = (S, O)$ be a topological space, and let $\boldsymbol{S} = (T, \boldsymbol{O})$ be a ringed space.

Restrictions of ringed spaces: Let U be an open set in O. We define the **restriction of $\boldsymbol{S}$ to** U, written $\boldsymbol{S}|_U$, to be the pair $(T|_U, \boldsymbol{O}|_U)$. The restriction $\boldsymbol{S}|_U$ is a ringed space (proof omitted).

Locally ringed spaces: $\boldsymbol{S}$ is called a **locally ringed space** if each of the stalks $\{\boldsymbol{O}_a\}_{a \in S}$ is a local ring.

Local homomorphisms of local rings: Let A be a local ring with maximal ideal $\mathbf{m}_A$, and let B be a local ring with maximal ideal $\mathbf{m}_B$. Let $\phi\colon A \to B$ be a homomorphism of rings. if $\phi(\mathbf{m}_A) = \mathbf{m}_B$, then we say that ϕ is a **local homomorphism**.

Morphisms of ringed spaces: Let $\boldsymbol{S} = (T, \boldsymbol{O})$ and $\boldsymbol{S}' = (T', \boldsymbol{O}')$ be ringed spaces. A **morphism** from $\boldsymbol{S}$ to $\boldsymbol{S}'$ is a pair $\Phi = (\phi, \psi)$ such that the following conditions hold:

1. We require that ϕ is a continuous map from $T = (S, O)$ to $T' = (S', O')$.
2. We require that ψ is a morphism of sheaves from $\boldsymbol{O}'$ to $\phi_*\boldsymbol{O}$. See § 2.3 for the definition of a morphism of sheaves and § 2.5 for the definition of $\phi_*\boldsymbol{O}$. Note that both $\boldsymbol{O}'$ and $\phi_*\boldsymbol{O}$ are sheaves of rings on T', so the definition of a morphism of sheaves given in § 2.3 applies.
3. For each element a in S, the map ψ induces a map $\psi_a\colon \boldsymbol{O}'_{\phi(a)} \to \boldsymbol{O}_a$.[6] If $\boldsymbol{S}$ and $\boldsymbol{S}'$ are locally ringed spaces, then we require that each map ψ_a be a local homomorphism.

If ϕ is a homeomorphism and ψ is an isomorphism, then we say that Φ is an **isomorphism**.

4. Affine Schemes

In this section we define the concept of an **affine scheme**, which is the analog in scheme theory of an affine variety. Recall that, in classical algebraic geometry, for each affine variety V over an algebraically closed field K, we define the coordinate ring $K[V]$. This is the ring $K[z]/I(V)$, where $I(V)$ is the ideal of polynomials in $K[z]$ that vanish on V. See *Definitions for Classical Algebraic Geometry*, § 7.1. In scheme theory, we go in the other direction: we start with a commutative ring R, and we define a ringed space called the **spectrum** of R, or Spec R.[7] The relationship between R and Spec R is analogous to the relationship between $K[V]$ and the ringed space formed by V together with its sheaf of regular functions (§ 2.1). Then we use Spec R to define the concept of an affine scheme.

This approach admits a larger class of "coordinate rings" than in classical algebraic geometry. For example, the "coordinate ring" doesn't have to be a K-algebra for an algebraically closed field K, and it doesn't have to be finitely generated. Also, there needn't be any coordinates, because we are not constrained to work in affine space $\mathbf{A}^n$.

---

[6] Fix an open set $U \subseteq S'$ containing $\phi(a)$. Then a is an element of $\phi^{-1}(U)$. Now fix an element y in $\boldsymbol{O}'(U)$ and consider the element $x = [y, U]_{\phi(a)}$ of $\boldsymbol{O}'_{\phi(a)}$. Then ψ_U is a map from $\boldsymbol{O}'(U)$ to $(\phi_*\boldsymbol{O})(U) = \boldsymbol{O}(\phi^{-1}(U))$. Therefore we may set $\psi_a(x) = [\psi_U(y), \phi^{-1}(U)]_a$.

[7] This notation agrees with [Eisenbud and Harris 1999]. [Hartshorne 1977] writes Spec R to denote the topological space of the ringed space, without the sheaf.

In the rest of this section, R denotes a commutative ring. We proceed as follows:

- In § 4.1, we define the topological space associated with Spec R.
- In § 4.2, we define the sheaf of regular functions of Spec R.
- In § 4.3, we define Spec R.
- In § 4.4, we give the definition of an affine scheme.

4.1. The Topological Space of Spec R

In this section we define the topological space $T = (S, O)$ associated with Spec R.

The set S: In § 1, we observed that the points a of $\mathbf{A}^n$ correspond to the maximal ideals $\mathbf{m}_a$ of $K[z]$. More generally, for any affine variety $V \subseteq \mathbf{A}^n$, the points a of V correspond to the maximal ideals $\mathbf{m}_a$ of the coordinate ring $K[V]$. Thus, in classical algebraic geometry, for a given coordinate ring R, the points in the Zariski topology are the maximal ideals of R.

In scheme theory, we adopt this point of view, and we generalize it. We define the points of S of Spec R to be the *prime* ideals of R. Because every maximal ideal is prime, this definition gives us all the points of classical algebraic geometry, plus some extra points. For example:

1. When R is the polynomial ring $K[z]$, S consists of the points of $\mathbf{A}^n$ (i.e., the maximal ideals $\mathbf{m}_a$) plus the irreducible affine varieties $V \subseteq \mathbf{A}_n$.
2. When R is the coordinate ring $K[V]$ of the affine variety $V \subseteq \mathbf{A}^n$, S consists of the points of V plus the irreducible affine subvarieties of V.

The open sets O: We define O analogously to the Zariski topology on $\mathbf{A}^n$, and again we call T the Zariski topology. We make the following constructions:

1. For any prime ideal $\mathbf{p}$ of R, let $K_\mathbf{p}$ denote the quotient field of the integral domain $R/\mathbf{p}$.
2. For any $r \in R$, and any prime ideal $\mathbf{p}$ of R, let $r(\mathbf{p})$ denote the element of $K_\mathbf{p}$ represented by r. In particular, when $r(\mathbf{p}) = 0$, we say that r **vanishes** at $\mathbf{p}$.

Note that these constructions generalize the ones we made in § 1, with $R = K[z]$. There we said that, for $p \in R$ and $\mathbf{m}$ a maximal ideal of R, $p(\mathbf{m})$ is the element of the field $R/\mathbf{m}$ represented by p.[8]

Next we define sets $V(F)$ as in § 1. Fix a family $F = \{r_\alpha\}$ of elements of R. Define $V(F)$ to be the zero set of F, i.e., the set of all prime ideals $\mathbf{p}$ of R such that $r_\alpha(\mathbf{p}) = 0$ for all r_α in F. Note that $r_\alpha(\mathbf{p}) = 0$ if and only if r_α is a member of the ideal $\mathbf{p}$, considered as a set of elements of R. Therefore $V(F)$ is the set of all prime ideals $\mathbf{p}$ such that $F \subseteq \mathbf{p}$. Note also that we use the same notation $V(F)$ as in § 1, but in the context of Spec R we don't call $V(F)$ a variety.

Now we define the Zariski topology exactly as in § 1: the open sets O are the complements of the closed sets, and the closed sets are the sets $V(F)$, where F is a family of elements of R. Setting $F = \{0\}$ shows that R is closed, so $\varnothing$ is open. Setting $F = \{r\}_{r \in R}$ shows that $\varnothing$ is closed, so R is open.[9] Otherwise, the proof that this definition gives a topology is the same one given in § 5.1 of *Definitions for Classical Algebraic Geometry*.

Let $\mathbf{p}$ be a point of S, i.e., a prime ideal of R. We assert the following (proofs omitted):

1. The closure of the set $\{\mathbf{p}\}$ consists of all prime ideals $\mathbf{q} \subseteq R$ such that $\mathbf{p} \subseteq \mathbf{q}$. In particular, the set $\{\mathbf{p}\}$ is closed if and only if $\mathbf{p}$ is a maximal ideal. When $\{\mathbf{p}\}$ is closed, we say that $\mathbf{p}$ is a **closed point**.
2. Assume that R is the coordinate ring $K[V]$ of an affine variety V over an algebraically closed field K. Then
   a. The points of V correspond to the closed points of S (i.e., the maximal ideals).
   b. Given a prime ideal $\mathbf{p} \subseteq R$, the closed points contained in the closure of $\{\mathbf{p}\}$ are exactly the points in the subvariety of V cut out by $\mathbf{p}$.

[8] Here is a motivation for using prime ideals as the points of schemes. By the observation we made in § 1, we can represent a point $\mathbf{p}$ as the set of all functions p such that $p(\mathbf{p}) = 0$. If $pq \in \mathbf{p}$, then $pq(\mathbf{p}) = p(\mathbf{p})q(\mathbf{p}) = 0$. Since p and q take their values in a field, and fields are integral domains, we must have $p(\mathbf{p}) = 0$ or $q(\mathbf{p}) = 0$, i.e., $p \in \mathbf{p}$ or $q \in \mathbf{p}$. So prime ideals are the maximally general points in this sense.

[9] Recall that a prime ideal $\mathbf{p}$ must be a proper subset of R. See *Definitions for Commutative Algebra*, § 6. Therefore there is no prime ideal $\mathbf{p}$ such that $r(\mathbf{p}) = 0$ for all r in R.

Distinguished open sets: Let $T = (S, O)$ be the topological space associated with Spec R, and let r be an element of R. We write S_r to denote the open set $R - V(r)$, where $V(r)$ is a shorthand for $V(\{r\})$. Note that $V(r)$ is closed, so its complement in R is open. Note also that S_r consists of exactly the prime ideals $\mathbf{p}$ of R such that $r \notin \mathbf{p}$. We call S_r the **distinguished open set** associated with r. This definition is analogous to the definition of a distinguished open set in $\mathbf{A}^n$ given in § 5 of *Definitions for Classical Algebraic Geometry*.

The distinguished open sets $\{S_r : r \in R\}$ form a base for the Zariski topology T. The proof of this fact is the same as the one given in § 5 of *Definitions for Classical Algebraic Geometry*.

The distinguished open sets S_r are closed under finite intersection (proof omitted).

4.2. The Sheaf of Regular Functions of Spec R

Now we define a sheaf of rings on T called the **sheaf of regular functions** of Spec R. This sheaf is also called the **structure sheaf** of Spec R. It is denoted $\boldsymbol{O}$ or $\boldsymbol{O}_{\text{Spec } R}$.

By the discussion in § 2.2, it suffices to define a B-sheaf on the distinguished open sets S_r of O. To do this, we need to do the following:

1. Define the set $\boldsymbol{O}(S_r)$ for each $r \in R$.
2. Define the restriction map $\boldsymbol{O}_{r,s} : \boldsymbol{O}_r \to \boldsymbol{O}_s$ for each pair (S_s, S_r) such that $S_s \subseteq S_r$.
3. Show that these definitions satisfy the sheaf axiom.

The sets $\boldsymbol{O}(S_r)$: For any $r \in R$, let X_r denote the multiplicative monoid $\{1, r, r^2, \dots\}$ consisting of all powers r^n with $n \geq 0$. Let $R[r^{-1}]$ denote the ring of fractions $X_r^{-1}R$. We may represent each element of this ring as p/r^n, for some $p \in R$ and $n \geq 0$. See *Definitions for Commutative Algebra*, § 14.

Let P_r be the set of prime ideals of $R[r^{-1}]$, and let $\phi : S_r \to P_r$ be the map that takes a prime ideal $\mathbf{p}$ in S_r to the prime ideal $\{p/x : p \in \mathbf{p}, x \in X\}$ in $R[r^{-1}]$. Then ϕ is a bijection (proof omitted). Thus we may identify the elements of S_r with the prime ideals of $R[r^{-1}]$. Accordingly, we make the following definition:

$$\boldsymbol{O}(S) = R$$

$$\boldsymbol{O}(S_r) = \boldsymbol{O}(S)[r^{-1}] = R[r^{-1}] \text{ for all } r \text{ in } R$$

Compare the observations in § 7.1 of *Definitions for Classical Algebraic Geometry*. There we said that the ring of regular functions on the distinguished open set $V - V(q)$ of an affine variety V is $K[V][q^{-1}]$. So the definitions agree in this case.

The restriction maps $\boldsymbol{O}_{r,s}$: Suppose $S_s \subseteq S_r$. Then $S_r^C \subseteq S_s^C$, so for any prime ideal $\mathbf{p}$ of R, if $r \in \mathbf{p}$, then $s \in \mathbf{p}$. Thus s is a member of the intersection of the prime ideals containing r, and this intersection equals the radical of the principal ideal (r). See *Definitions for Commutative Algebra*, § 6. Therefore $s^n = pr$ for some $n \geq 0$ and $p \in R$.

Now consider an element of $R[r^{-1}]$. Any such element has a representation q/r^m, with $q \in R$ and $m \geq 0$. Define

$$\boldsymbol{O}_{r,s}(q/r^m) = qp^m/s^{mn} \text{ where } s^n = pr$$

Then the right-hand side is an element of $\boldsymbol{O}(S_s) = R[s^{-1}]$, as required. Further, plugging in $s^n = pr$ on the right-hand side and simplifying terms yields q/r^m.

The sheaf axiom: For the proof that this definition satisfies the sheaf axiom, see [Eisenbud and Harris 1999], § 1.1.4.

4.3. The Definition of Spec R

For any commutative ring R, we define Spec R to be the pair $(T, \boldsymbol{O})$, where $T = (S, O)$ is the topological space defined in § 4.1, and $\boldsymbol{O}$ is the sheaf of regular functions on T defined in § 4.2. In § 5.1, we will see the way in which the elements of the rings $\boldsymbol{O}(U)$ may be interpreted as functions.

Spec R is a locally ringed space (§ 3) (proof omitted).

4.4. The Definition of an Affine Scheme

Now we can define an affine scheme. Let $T = (S, O)$ be a topological space. An **affine scheme** is a locally ringed space $\boldsymbol{S} = (T, \boldsymbol{O})$ (§ 3) that is isomorphic to the locally ringed space Spec $\boldsymbol{O}(S)$ (§ 4.3).

5. Schemes

We now have all the machinery we need to define general schemes.

5.1. The Definition of a Scheme

A **scheme** is a locally ringed space $\boldsymbol{S} = (T, \boldsymbol{O})$ such that $\boldsymbol{S}$ is **locally affine**. This means there is an open cover $\{U_i\}$ of $\boldsymbol{S}$ such that, for each U_i, the restriction $\boldsymbol{S}|_{U_i}$ of $\boldsymbol{S}$ to U_i (§ 3) is the affine scheme

$$\boldsymbol{S}_i = (T_i, \boldsymbol{O}_i) = ((S_i, O_i), \boldsymbol{O}_i) = \text{Spec } \boldsymbol{O}_i(S_i)$$

(§ 4.4). We say that the affine schemes $\boldsymbol{S}_i$ **cover** the scheme $\boldsymbol{S}$. Compare the definition of a variety V in projective space, which is covered by affine open sets $U_i \cap V$. See *Definitions for Classical Algebraic Geometry*, § 6.2.

Let $\boldsymbol{S} = (T, \boldsymbol{O}) = ((S, O), \boldsymbol{O})$ be a scheme.

1. The stalks $\{\boldsymbol{O}_a : a \in S\}$ are called the **local rings** of $\boldsymbol{S}$.
2. For any $a \in S$, we denote by K_a the residue field of $\boldsymbol{O}_a$, i.e., the ring $\boldsymbol{O}_a$ modulo its unique maximal ideal. Let $\phi : \boldsymbol{O}_a \to K_a$ be the natural map, i.e., the map that takes each element of $\boldsymbol{O}_a$ to the equivalence class that it represents in K_a. Let D be the disjoint union $\biguplus_{a \in S} K_a$, i.e., the set $\{(a, k) : a \in S, k \in K_a\}$.
3. Let U be an open set of O. A **regular function** on U is an element p of $\boldsymbol{O}(U)$. We may associate with any such element the function $f_p : U \to D$ given by $f_p(a) = (a, \phi([U, p]_a))$. Note the similarity to the construction that we used for the presheaf of stalk maps (§ 2.2).
4. A **global regular function** is a regular function on S.
5. $\boldsymbol{S}$ is **irreducible** if S cannot be expressed as the union of two closed sets, each properly contained in S (i.e., neither equal to S).
6. Let $\{U_i\}_{i \in I}$ be an open cover of S such that each $\boldsymbol{S}|_{U_i}$ is Spec R_i. If I is a finite set and each R_i is a Noetherian ring, then we say that $\boldsymbol{S}$ is **Noetherian**.

5.2. Open Subschemes

Let $\boldsymbol{S} = (T, \boldsymbol{O}) = ((S, O), \boldsymbol{O})$ be a scheme. Let $U \subseteq S$ be an open subset. The restriction $\boldsymbol{S}|_U$ is a scheme (proof omitted). We say that $\boldsymbol{S}|_U$ is an **open subscheme** of $\boldsymbol{S}$. There is one open subscheme $\boldsymbol{S}|_U$ of $\boldsymbol{S}$ for each open set $U \subseteq S$.

Let $\boldsymbol{S}|_U$ be an open subscheme of $\boldsymbol{S}$. If $\boldsymbol{S}|_U$ is an affine scheme, then we say that it is an **affine open subscheme** of $\boldsymbol{S}$, and we say that U is an **affine open set** in the topology T. By definition, $\boldsymbol{S}$ is covered by affine open subschemes $\boldsymbol{S}|_{U_i}$, and S is covered by affine open sets U_i (§ 5.1).

5.3. Sheaves of *O*-Modules

Let $\boldsymbol{S} = (T, \boldsymbol{O}) = ((S, O), \boldsymbol{O})$ be a scheme, and let $\boldsymbol{F}$ be a sheaf of abelian groups on T such that the following facts are true:

1. For each open set $U \subseteq S$, $\boldsymbol{F}(U)$ is an $\boldsymbol{O}(U)$-module.
2. For each pair of open sets $V \subseteq U \subseteq S$, and for any a in $\boldsymbol{O}(U)$ and b in $\boldsymbol{F}(U)$, we have

$$\boldsymbol{F}_{U,V}(ab) = \boldsymbol{O}_{U,V}(a)\boldsymbol{F}_{U,V}(b).$$

In this case we say that $\boldsymbol{F}$ is a **sheaf of *O*-modules** on $\boldsymbol{S}$.

Quasicoherent sheaves of *O*-modules: We now define a concept called a quasicoherent sheaf of $\boldsymbol{O}$-modules. We will use this concept to define closed subschemes (§ 5.6).

Let $\boldsymbol{S} = ((S, O), \boldsymbol{O})$ be a scheme, and let $\boldsymbol{F}$ be a sheaf of $\boldsymbol{O}$-modules on $\boldsymbol{S}$.

1. Assume that $\boldsymbol{S}$ is the affine scheme Spec $\boldsymbol{O}(S)$. We say that $\boldsymbol{F}$ is **quasicoherent** if, for each distinguished open set $S_r \subseteq S$, we have

$$\boldsymbol{F}(S_r) \cong \boldsymbol{F}(S)[r^{-1}].$$

Note that $\boldsymbol{F}(S)$ is an $\boldsymbol{O}(S)$-module, so $\boldsymbol{F}(S_r)$ is an $\boldsymbol{O}(S)[r^{-1}]$-module, as required. See *Definitions for Commutative Algebra*, § 14. Note also that $\boldsymbol{O}$ itself is a quasicoherent sheaf of $\boldsymbol{O}$-modules on $\boldsymbol{S}$, because $\boldsymbol{O}(S_r) = \boldsymbol{O}(S)[r^{-1}]$. See § 4.2.

2. Now let $\boldsymbol{S}$ be a general scheme. For every open set $U \subseteq S$, $\boldsymbol{F}|_U$ is a sheaf of $\boldsymbol{O}$-modules on the scheme $\boldsymbol{S}|_U$. We say that $\boldsymbol{F}$ is **quasicoherent** if, for every affine open subscheme $\boldsymbol{S}|_U$ of $\boldsymbol{S}$, $\boldsymbol{F}|_U$ is quasicoherent in the sense of item 1.

Coherent sheaves of $\boldsymbol{O}$-modules: Let $\boldsymbol{S}$ be a scheme, and let $\boldsymbol{F}$ be a sheaf of $\boldsymbol{O}$-modules on $\boldsymbol{S}$. We say that $\boldsymbol{F}$ is **coherent** if it is quasicoherent and all the modules $\boldsymbol{F}(U)$ are finitely generated.

5.4. Sheaves of $\boldsymbol{O}$-Algebras

Fix a ring A. Recall that an A-algebra is a ring B that is also an A-module, such that the map $a \mapsto a \cdot 1_B$ is a ring homomorphism. See *Definitions for Commutative Algebra*, § 9. Let $\boldsymbol{F}$ be a sheaf of $\boldsymbol{O}$-modules on $\boldsymbol{S}$ (§ 5.3). If each module $\boldsymbol{F}(U)$ is an $\boldsymbol{O}(U)$-algebra, then we say that $\boldsymbol{F}$ is a **sheaf of $\boldsymbol{O}$-algebras on $\boldsymbol{S}$.** A sheaf of $\boldsymbol{O}$-algebras is quasicoherent if it is quasicoherent as a sheaf of $\boldsymbol{O}$-modules; and similarly for coherent sheaves.

5.5. Sheaves of Ideals

In this section, let $\boldsymbol{S} = (T, \boldsymbol{O}) = ((S, O), \boldsymbol{O})$ be a scheme.

Sheaves of ideals on $\boldsymbol{S}$: Let $\boldsymbol{I}$ be a sheaf of $\boldsymbol{O}$-modules on $\boldsymbol{S}$ (§ 5.3) such that, for every open set $U \subseteq S$, $\boldsymbol{I}(U) \subseteq \boldsymbol{O}(U)$. In this case $\boldsymbol{I}(U)$ is an ideal of $\boldsymbol{O}(U)$, and we say that $\boldsymbol{I}$ is a **sheaf of ideals on $\boldsymbol{S}$**.

Sheaves of ideals in $\boldsymbol{F}$: Fix a sheaf $\boldsymbol{F}$ of $\boldsymbol{O}$-algebras on $\boldsymbol{S}$. Let $\boldsymbol{I}$ be a sheaf of $\boldsymbol{O}$-modules on $\boldsymbol{S}$ such that for every open set $U \subseteq S$, $\boldsymbol{I}(U) \subseteq \boldsymbol{F}(U)$. In this case $\boldsymbol{I}(U)$ is an ideal of $\boldsymbol{F}(U)$, and we say that $\boldsymbol{I}$ is a **sheaf of ideals in $\boldsymbol{F}$**. Note that a sheaf of ideals on $\boldsymbol{S} = (T, \boldsymbol{O})$ is a sheaf of ideals in $\boldsymbol{O}$.

Quasicoherent sheaves of ideals: Let $\boldsymbol{S}$ be a scheme, and let $\boldsymbol{I}$ be a sheaf of ideals on $\boldsymbol{S}$ that is quasicoherent as a sheaf of $\boldsymbol{O}$-modules (§ 5.3). Then by the definition of a quasicoherent sheaf of modules, we have the following:

1. Assume that $\boldsymbol{S}$ is an affine scheme, and let I denote the ideal $\boldsymbol{I}(S) \subseteq \boldsymbol{O}(S)$. For all distinguished open sets S_r we have the $\boldsymbol{O}(S_r)[r^{-1}]$-module isomorphism

$$\boldsymbol{I}(S_r) \cong I[r^{-1}]$$

2. In general, property 1 holds for all affine open subschemes $\boldsymbol{S}|_U$.

Prime ideal sheaves: Let $\boldsymbol{F}$ be a quasicoherent sheaf of $\boldsymbol{O}$-algebras on $\boldsymbol{S}$ (§ 5.4). Let $\boldsymbol{I}$ be a sheaf of ideals in $\boldsymbol{F}$ that is quasicoherent as a sheaf of $\boldsymbol{O}$-modules on $\boldsymbol{S}$. Assume the following:

1. For each open set U in S, either (a) $\boldsymbol{I}(U)$ is a prime ideal of $\boldsymbol{F}(U)$ or (b) $\boldsymbol{I}(U) = \boldsymbol{F}(U)$.
2. $\boldsymbol{I} \neq \boldsymbol{F}$, i.e., for at least one open set U of S, $\boldsymbol{I}(U) \neq \boldsymbol{F}(U)$.

In this case we say that $\boldsymbol{I}$ is a **prime ideal sheaf in $\boldsymbol{F}$**.

Proposition: Fix an affine scheme $\boldsymbol{S} = ((S, O), \boldsymbol{O}) = \operatorname{Spec} \boldsymbol{O}(S)$. The points of S are in one-to-one correspondence with the prime ideal sheaves in $\boldsymbol{O}$.

Proof: Observe that $\boldsymbol{O}$ is a quasicoherent sheaf of $\boldsymbol{O}$-algebras on $\boldsymbol{S}$ (§ 5.4). Let P be the set of prime ideal sheaves in $\boldsymbol{O}$. We must establish a bijection between P and S.

1. Because $\boldsymbol{I}$ is quasicoherent, it is determined by the ideal $I = \boldsymbol{I}(S) \subseteq \boldsymbol{O}(S)$ (§ 5.5). If $I = \boldsymbol{O}(S)$, then $\boldsymbol{I}(S_r) = \boldsymbol{O}(S)[r^{-1}] = \boldsymbol{O}(S_r)$ for all distinguished open sets S_r in S, so $\boldsymbol{I} = \boldsymbol{O}$, and $\boldsymbol{I}$ is not a prime ideal sheaf. Therefore I must be a prime ideal of $\boldsymbol{O}(S)$, i.e., a point of S. Each ideal corresponds to a different prime ideal sheaf, so there is an injective map from P to S.
2. By the definition of a quasicoherent sheaf of ideals, each prime ideal $I \subseteq \boldsymbol{O}(S)$ determines a different prime ideal sheaf in P. Thus there is an injective map from S to P.

□

5.6. Closed Subschemes

Affine schemes: Fix a ring R, and let $\boldsymbol{S} = (T, \boldsymbol{O}) = ((S, O), \boldsymbol{O})$ be the affine scheme Spec R, so that S is the set of prime ideals of R. Let $I \subseteq R$ be an ideal. Observe the following:

1. The closed set $V(I)$ is the set of all prime ideals $\mathbf{p} \subseteq R$ such that $I \subseteq \mathbf{p}$ (§ 4.1).
2. The prime ideals of R/I are exactly the prime ideals $\mathbf{p} \subseteq R$ such that $I \subseteq \mathbf{p}$, taken modulo I.

Therefore we identify Spec R/I with $V(I)$ and call Spec R/I a **closed subscheme** of Spec R. There is one closed subscheme R/I of Spec R for each ideal $I \subseteq R$. We say that I is the ideal of R **associated to** the closed subscheme R/I.

General schemes: Let $\boldsymbol{S} = (T, \boldsymbol{O}) = ((S, O), \boldsymbol{O})$ be a scheme, and let X be a closed subset of S, i.e., $X^C \in O$. Let $T' = (X, O')$ be a topological space, and let $\boldsymbol{O}'$ be a sheaf of rings on T'. We say that $\boldsymbol{S}' = (T', \boldsymbol{O}')$ is a **closed subscheme** of $\boldsymbol{S}$ if the following statements are true:

1. There exists a quasicoherent sheaf of ideals $\boldsymbol{I}$ on $\boldsymbol{S}$ (§ 5.5) such that for each affine subscheme $\boldsymbol{S}|_U$, $X \cap U$ is the set of points of the closed subscheme of $\boldsymbol{S}|_U$ associated to the ideal $\boldsymbol{I}(U) \subseteq \boldsymbol{O}(U)$.
2. Let $\phi \colon X \to S$ be the inclusion map. The pushforward $\phi_* \boldsymbol{O}'$ is the quotient sheaf $\boldsymbol{O}/\boldsymbol{I}$, interpreting $\boldsymbol{O}$ and $\boldsymbol{O}'$ as sheaves of abelian groups.

Note the following (proofs omitted):

1. When $\boldsymbol{S}$ is an affine scheme Spec R, there are two definitions of a closed subscheme of $\boldsymbol{S}$ (with $\boldsymbol{S}$ as an affine scheme and with $\boldsymbol{S}$ as a general scheme), and these definitions agree. Let I be an ideal of R, so that Spec R/I is a closed subscheme of $\boldsymbol{S}$. The quasicoherent sheaf of ideals in the general definition is $\boldsymbol{I}(U) = I\ \boldsymbol{O}(U)$.
2. Each quasicoherent sheaf of ideals $\boldsymbol{I}$ on $\boldsymbol{S}$ induces one and only one closed subscheme $\boldsymbol{S}_{\boldsymbol{I}}$ of $\boldsymbol{S}$. We say that $\boldsymbol{I}$ is the quasicoherent sheaf of ideals **associated to** the closed subscheme $\boldsymbol{S}_{\boldsymbol{I}}$.

Fix a topological space $T = (S, O)$, a scheme $\boldsymbol{S} = (T, \boldsymbol{O})$, and a closed subscheme $\boldsymbol{S}_{\boldsymbol{I}}$ of $\boldsymbol{S}$. Let V be an open subset of O, and let $f \in \boldsymbol{O}(V)$ be a regular function. We say that f **vanishes** on $\boldsymbol{S}'$ if f is an element of $\boldsymbol{I}(V)$.

5.7. Locally Closed Subschemes

Let $\boldsymbol{S} = (T, \boldsymbol{O})$ be a scheme, let U be an open subset of T, and let $\boldsymbol{S}|_U$ be the corresponding open subscheme of $\boldsymbol{S}$ (§ 5.2). A closed subscheme of $\boldsymbol{S}|_U$ is called a **locally closed subscheme** of $\boldsymbol{S}$.

Now let $\boldsymbol{S}$ be a scheme, and let $\boldsymbol{S}'$ be a locally closed subscheme of $\boldsymbol{S}$. The **closure** of $\boldsymbol{S}$, written $\overline{\boldsymbol{S}}$, is the smallest closed subscheme of $\boldsymbol{S}$ containing $\boldsymbol{S}'$. The following definition is equivalent (proof omitted):

1. Assume that $\boldsymbol{S}'$ is a closed subscheme of the open subscheme $\boldsymbol{S}|_U$ of $\boldsymbol{S}$.
2. Let $\boldsymbol{F}$ be the sheaf of ideals on $\boldsymbol{S}$ defined as follows: $\boldsymbol{F}(W)$ is the set of regular functions in $\boldsymbol{O}(W)$ whose restrictions to W vanish on U.
3. Then $\overline{\boldsymbol{S}'}$ is the closed subscheme $\boldsymbol{S}_{\boldsymbol{F}}$ of $\boldsymbol{S}$ (§ 5.6).

5.8. Reduced Schemes

In this section we define the concept of a reduced scheme. Despite what the names suggest, this concept is unrelated to the concept of an irreducible scheme (§ 5.1).

Affine schemes: Let R be a ring. Recall that an element r of R is **nilpotent** if there exists an integer $n > 0$ such that $r^n = 0$. Recall that the set of all nilpotent elements of R is an ideal, called the **nilradical** of R, and that it is the intersection of the prime ideals of R. See *Definitions for Commutative Algebra*, § 6. Let N denote the nilradical of R.

Now consider the affine scheme $\boldsymbol{S} = \text{Spec } R$. We define the **reduced scheme associated to** $\boldsymbol{S}$, denoted $\boldsymbol{S}_{\text{red}}$, to be Spec R/N, that is, the spectrum of R modulo its nilradical. Note the following:

1. $\boldsymbol{S}_{\text{red}}$ is a closed subscheme of $\boldsymbol{S}$ (§ 5.6).
2. Let $\boldsymbol{S} = (T, \boldsymbol{O})$ and $\boldsymbol{S}_{\text{red}} = (T_{\text{red}}, \boldsymbol{O}_{\text{red}})$. Then $T = T_{\text{red}}$ as topological spaces (proof omitted).
3. $(\boldsymbol{S}_{\text{red}})_{\text{red}} = \boldsymbol{S}_{\text{red}}$.

We say that an affine scheme $\boldsymbol{S}$ is **reduced** if $\boldsymbol{S} = \boldsymbol{S}_{\text{red}}$. In particular, $\boldsymbol{S}_{\text{red}}$ is reduced.

General schemes: Let $\boldsymbol{S} = (T, \boldsymbol{O})$ be a general scheme. We define the **nilradical** of $\boldsymbol{S}$ to be the sheaf of ideals N on $\boldsymbol{S}$ such that, for each open set U of T, $N(U)$ is the nilradical of $\boldsymbol{O}(U)$. N is a quasicoherent sheaf of ideals on $\boldsymbol{S}$ (§ 5.5) (proof omitted).

Let $\boldsymbol{S}_{\text{red}}$ be the closed subscheme of $\boldsymbol{S}$ associated to N (§ 5.6). We say that $\boldsymbol{S}_{\text{red}}$ is the **reduced scheme associated to** $\boldsymbol{S}$. Again we have $(\boldsymbol{S}_{\text{red}})_{\text{red}} = \boldsymbol{S}_{\text{red}}$, and we say that $\boldsymbol{S}$ is **reduced** if $\boldsymbol{S} = \boldsymbol{S}_{\text{red}}$.

Let $\boldsymbol{S}$ be an affine scheme. Then the definition of $\boldsymbol{S}_{\text{red}}$ given for $\boldsymbol{S}$ as a general scheme agrees with the definition of $\boldsymbol{S}_{\text{red}}$ given for $\boldsymbol{S}$ as an affine scheme (proof omitted).

5.9. The Local Ring at a Point

Let $\boldsymbol{S} = (T, \boldsymbol{O}) = ((S, O), \boldsymbol{O})$ be a scheme. As noted in § 5.1, for each point a in S the stalk $\boldsymbol{O}_a$ is called the **local ring** of $\boldsymbol{S}$ at a. In this section we state some geometric definitions associated with these local rings.

The maximal ideal: For each point a of S, let $\mathbf{m}_a$ be the unique maximal ideal of the local ring $\boldsymbol{O}_a$. $\mathbf{m}_a$ is the set of all elements $[U, p]_a$ such that U is an element of O, U contains a, p is an element of $\boldsymbol{O}(U)$, and p vanishes at a (§ 4.1) (proof omitted).

The dimension: We define the **dimension** of $\boldsymbol{S}$ as follows:

1. The dimension of $\boldsymbol{S}$ at a, denoted $\dim_a \boldsymbol{S}$, is the Krull dimension of $\boldsymbol{O}_a$, i.e., the supremum of the lengths of all chains of prime ideals of $\boldsymbol{O}_a$.
2. The dimension of $\boldsymbol{S}$, denoted $\dim \boldsymbol{S}$, is the supremum of the dimensions $\dim_a \boldsymbol{S}$ at the points a in S.

Notice that this definition generalizes the dimension of a variety, as defined in § 20 of *Definitions for Classical Algebraic Geometry*. There we said that for an irreducible variety V, (1) the local rings $\boldsymbol{O}_a$ all have the same dimension; and (2) for each a in V, $\dim \boldsymbol{O}_a = \dim V$. We also said that for a general variety V, the dimension is the maximum of the dimensions of the (finitely many) irreducible components of V.

The Zariski tangent space: Let a be a point of S. We define the **Zariski tangent space** to $\boldsymbol{S}$ at a, denoted T_a, exactly as in § 15.1 of *Definitions for Classical Algebraic Geometry*:

1. Let F_a be the field $\boldsymbol{O}_a/\mathbf{m}_a$. The **Zariski cotangent space** to $\boldsymbol{S}$ at a is the F_a-vector space $T_a^* = \mathbf{m}_a/\mathbf{m}_a^2$.
2. The Zariski tangent space T_a is the dual space T_a^{**}, i.e., the space of linear maps $\lambda\colon T_a^* \to F_a$.

Singular and nonsingular points: Let a be a point of S.

1. If $\dim T_a = \dim_a \boldsymbol{S}$, then we say that $\boldsymbol{S}$ is **nonsingular** or **regular** at a.
2. Otherwise $\dim T_a \geq \dim_a \boldsymbol{S}$, and we say that $\boldsymbol{S}$ is **singular** at a.

When $\boldsymbol{S}$ is Noetherian (§ 5.1), then by definition $\boldsymbol{S}$ is nonsingular at a point a if and only if $\boldsymbol{O}_a$ is a regular local ring. See *Definitions for Commutative Algebra*, § 31.

5.10. Morphisms

In this section, let $\boldsymbol{S} = (T, \boldsymbol{O})$ and $\boldsymbol{S}' = (T', \boldsymbol{O}')$ be schemes, with $T = (S, O)$ and $T' = (S', O')$.

General morphisms: Recall that $\boldsymbol{S}$ and $\boldsymbol{S}'$ are locally ringed spaces that are locally affine (§ 5.1). A **scheme morphism** or **morphism** from $\boldsymbol{S}$ to $\boldsymbol{S}'$ is a morphism $\Phi = (\phi, \psi)$ of the locally ringed spaces (§ 3). Each scheme morphism satisfies the following condition (proof omitted):

> *Proposition: Let a be a point of S, and let U be an open neighborhood of $\phi(a)$ in S'. Let p be an element of $\boldsymbol{O}'(U)$. Then p vanishes at $\phi(a)$ if and only if $\psi_U(p)$ vanishes at a.*

Note that $\psi = \{\psi_U\}_{U \in O'}$ is a morphism of sheaves from $\boldsymbol{O}'$ to $\phi_*\boldsymbol{O}$ (§ 2.3). Therefore ψ_U is a ring homomorphism from $\boldsymbol{O}(U)$ to $\phi_*\boldsymbol{O}(U) = \boldsymbol{O}(\phi^{-1}(U))$ (§ 2.5).

Morphisms to affine schemes: Let R' be a ring, and assume that $\boldsymbol{S}'$ is the affine scheme Spec R'. Let $\Phi\colon \boldsymbol{S} \to \boldsymbol{S}'$ be a morphism, with $\Phi = (\phi, \psi)$. Note the following:

1. $\psi_{S'}$ is a ring homomorphism from $\boldsymbol{O}(S')$ to $\phi_*\boldsymbol{O}(S')$.
2. Because $\boldsymbol{S}' = \text{Spec } R'$, $\boldsymbol{O}(S') = R'$ (§ 4.2).
3. $\phi_*\boldsymbol{O}(S') = \boldsymbol{O}(\phi^{-1}(S')) = \boldsymbol{O}(S)$.

Therefore $\psi_{S'}$ is a ring homomorphism from R' to $\boldsymbol{O}(S)$.

Let hom($\boldsymbol{S}$, Spec R') denote the set of scheme morphisms from $\boldsymbol{S}$ to Spec R'. Let hom(R', $\boldsymbol{O}(S)$) denote the set of ring homomorphisms from R' to $\boldsymbol{O}(S)$. The following result establishes a one-to-one correspondence between these sets (proof omitted):

> *Theorem. The map* $(\phi, \psi) \mapsto \psi_{S'}$ *is a bijection between* hom($\boldsymbol{S}$, Spec R') *and* hom(R', $\boldsymbol{O}(S)$).

In particular, when $\boldsymbol{S}$ is the affine scheme Spec R, we have the following:

> *Corollary. The sets* hom(Spec R, Spec R') *and* hom(R', R) *are in one-to-one correspondence.*

Therefore the category of affine schemes is the category of commutative rings with arrows reversed.[10] Here is the corresponding statement from classical algebraic geometry:

> *Let V and W be affine varieties embedded in $\mathbf{A}^n(K)$. Let $\phi: V \to W$ denote any regular map, and let p denote any element of the coordinate ring $K[W]$. The map $\phi \mapsto (p \mapsto p \circ \phi)$ establishes a bijection between the regular maps $\phi: V \to W$ and the ring homomorphisms $\psi: K[W] \to K[V]$.*

Restrictions of morphisms: Let $\boldsymbol{S}$ and $\boldsymbol{S}'$ be schemes, let $\Phi = (\phi, \psi): \boldsymbol{S} \to \boldsymbol{S}'$ be a morphism, and let $\boldsymbol{S}_U$ be an open subscheme of $\boldsymbol{S}$ (§ 5.2). We define the **restriction** of Φ to $\boldsymbol{S}_U$, written $\Phi|_{\boldsymbol{S}_U}$ or Φ_U, as follows:

$$\Phi|_{\boldsymbol{S}_U} = (\phi|_U, \psi|_U).$$

6. Constructions on Schemes

We now describe several useful constructions on schemes.

6.1. The Gluing Construction

In this section we describe a construction called the **gluing construction** for assembling larger schemes from families of smaller schemes.

Gluing sheaves: First we describe a construction for gluing sheaves. Let $T = (S, O)$ be a topological space, and let $\{U_i\}_{i \in I}$ be an open cover of S. For each i, let $T_i = (U_i, O_i)$ be the topological space induced on U_i by the subset topology, and assume there is a sheaf $\boldsymbol{F}_i$ from T_i to $C = (M, \mu)$.

Under certain conditions we can combine or "glue" the sheaves $\boldsymbol{F}_i$ into a single sheaf $\boldsymbol{F}$ from T to C. For each $(i, j) \in I \times I$, let U_{ij} denote the set $U_i \cap U_j$. (Note that $U_{ij} = U_{ji}$.) For each triple (i, j, k) define the following sheaves:

- $\boldsymbol{F}_{ij} = \boldsymbol{F}_i|_{U_{ij}}$.
- $\boldsymbol{F}_{ijk} = \boldsymbol{F}_{ij}|_{U_k} = \boldsymbol{F}_i|_{U_{ij} \cap U_{ik}}$.

The conditions are as follows:

G1. For each $\boldsymbol{F}_{ij}$ there is a sheaf isomorphism $\phi_{ij}: \boldsymbol{F}_{ij} \to \boldsymbol{F}_{ji}$, such that ϕ_{ii} is the identity map.

G2. For each $\boldsymbol{F}_{ijk}$ we have $\phi_{jk} \circ (\phi_{ij}|_{\boldsymbol{F}_{ijk}}) = \phi_{ik}|_{\boldsymbol{F}_{ijk}}$

Under these conditions, there is a unique sheaf $\boldsymbol{F}$ from T to C with the following properties (proof omitted):

1. For each i there is a sheaf isomorphism $\psi_i: \boldsymbol{F}|_{U_i} \to \boldsymbol{F}_i$.
2. For each ϕ_{ij} we have $\phi_{ij} = \psi_j \circ (\psi_i^{-1}|_{\boldsymbol{F}_{ij}})$.

Gluing schemes (special case): We can use the gluing construction on sheaves to glue together affine schemes whose topological spaces are embedded in a common space. Again let $T = (S, O)$ be a topological space, and let $\{U_i\}_{i \in I}$ be an open cover of S. For each i, let $T_i = (U_i, O_i)$ be the topological space induced on U_i by the subset topology, and assume there is a sheaf of rings $\boldsymbol{O}_i$ on T_i that makes $\boldsymbol{S}_i = (T_i, \boldsymbol{O}_i)$ into an affine scheme, i.e., a locally ringed space that is isomorphic to Spec $\boldsymbol{O}_i(U_i)$ (§ 4.4). Assume further that the family $\{\boldsymbol{O}_i\}$ satisfies conditions **G1** and **G2** stated above. Then the gluing construction on $\{\boldsymbol{O}_i\}$ yields a sheaf $\boldsymbol{O}$ on T. Further, $\boldsymbol{S} = (T, \boldsymbol{O})$ is a scheme, because it is locally affine with respect to the open cover $\{U_i\}$ of T (§ 5.1).

Conversely, let $\boldsymbol{S} = (T, \boldsymbol{O})$ be a scheme covered by affine schemes $\{\boldsymbol{S}_i\}$. Then $\boldsymbol{S}$ is formed by gluing together the affine schemes $\boldsymbol{S}_i$ in this way. In this case we have $\boldsymbol{O}_i = \boldsymbol{O}|_{U_i}$ for each i, so $\boldsymbol{O}_i|_{U_{ij}} = \boldsymbol{O}_j|_{U_{ij}} = \boldsymbol{O}|_{U_{ij}}$, and the sheaf

[10] That is, the opposite category. See *Definitions for Category Theory*, § 8.

isomorphisms ϕ_{ij} required are the identity maps.

Gluing schemes (general case): We now describe a general construction for gluing schemes together. Let $\{\boldsymbol{S}_i\}_{i \in I}$ be a family of schemes, with $\boldsymbol{S}_i = (T_i, \boldsymbol{O}_i)$ and $T_i = (S_i, O_i)$. For each (i, j) in $I \times I$, let U_{ij} be an open subset of S_i such that $U_{ii} = S_i$. For all triples (i, j, k) define the following open subschemes of $\boldsymbol{S}$:

- $\boldsymbol{S}_{ij} = \boldsymbol{S}_i|_{U_{ij}}$.
- $\boldsymbol{S}_{ijk} = \boldsymbol{S}_{ij}|_{U_{ik}} = \boldsymbol{S}_i|_{U_{ij} \cap U_{ik}}$.

Suppose that these schemes satisfy the following conditions:

G3. For each $\boldsymbol{S}_{ij}$ there is a scheme isomorphism $\Phi_{ij}\colon \boldsymbol{S}_{ij} \to \boldsymbol{S}_{ji}$, such that Φ_{ii} is the identity map.

G4. For each $\boldsymbol{S}_{ijk}$ we have $\Phi_{jk} \circ (\Phi_{ij}|_{\boldsymbol{S}_{ijk}}) = \Phi_{ik}|_{\boldsymbol{S}_{ijk}}$.

Under these conditions, there is a unique scheme $\boldsymbol{S}$ with the following properties (proof omitted):

1. $\boldsymbol{S}$ is covered by affine schemes $\{\boldsymbol{S}'_i\}_{i \in I}$.
2. For each i there is a scheme isomorphism $\Psi_i\colon \boldsymbol{S}'_i \to \boldsymbol{S}_i$.
3. For each Φ_{ij} we have $\Phi_{ij} = \Psi_j \circ (\Psi_i^{-1}|_{\boldsymbol{S}_{ij}})$.

Notice that this construction generalizes the gluing construction for sheaves.

The affine space over a scheme: Here is a basic example of the gluing construction for schemes. For any scheme $\boldsymbol{S}$ and $n > 0$ we define an associated scheme called the n-dimensional **affine space** over $\boldsymbol{S}$, denoted $\mathbf{A}^n_{\boldsymbol{S}}$.

First, assume that $\boldsymbol{S}$ is isomorphic to Spec R for some ring R. Let $R[x]$ denote $R[x_1, \ldots, x_{n,}]$, i.e., the polynomial ring in n formal variables over R. We define $\mathbf{A}^n_{\boldsymbol{S}}$ to be the scheme Spec $R[x]$, i.e., the spectrum of the polynomial ring in n formal variables over R. We also denote this scheme $\mathbf{A}^n_R$. Notice that when R is an algebraically closed field K, we have $\mathbf{A}^n_K = \text{Spec } K[z]$. Let $T(\mathbf{A}^n_K)$ denote the topological space of $\mathbf{A}^n_K$. By definition this is the set of *prime* ideals of $K[z]$ (§ 4.1); whereas the the n-dimensional affine space $\mathbf{A}^n(K)$ of classical algebraic geometry consists of the *maximal* ideals of $K[z]$. So $T(\mathbf{A}^n_K)$ has $\mathbf{A}^n(K)$ embedded in it.

Now suppose that $\boldsymbol{S}$ is a scheme covered by affine schemes $\{\boldsymbol{S}_i\}$, where $\boldsymbol{S}_i = (T_i, \boldsymbol{O}_i) \cong \text{Spec } R_i$, and $T_i = (U_i, O_i)$. Let $T(\text{Spec } R_i)$ denote the topological space of Spec R_i. For each pair (i, j), let $U_{ij} = U_i \cap U_j$, and let $V_{ij} \subseteq T(\text{Spec } R_i)$ be the image of U_{ij} under the isomorphism $\boldsymbol{S}_i \cong \text{Spec } R_i$. Then by assumption there are scheme isomorphisms

$$\text{Spec } R_i|_{V_{ij}} \cong \text{Spec } R_j|_{V_{ji}}. \tag{1}$$

Let $T(\mathbf{A}^n_{R_i})$ denote the topological space of $\mathbf{A}^n_{R_i}$. Let $W_{ij} \subseteq T(\mathbf{A}^n_{R_i}) = T(\text{Spec } R_i[x])$ be the image of V_{ij} under the map $p \mapsto pR_i[x]$; it is straightforward to show that this map takes prime ideals of R_i to prime ideals of $R_i[x]$, and that W_{ij} is homeomorphic to V_{ij} and therefore an open subset of $T(\mathbf{A}^n_{R_i})$. Each scheme isomorphism (1) induces a scheme isomorphism

$$\mathbf{A}^n_{R_i}|_{W_{ij}} \cong \mathbf{A}^n_{R_j}|_{W_{ji}}. \tag{2}$$

The scheme isomorphisms (2) satisfy the gluing construction for the family $\{\mathbf{A}^n_{R_i}\}_{i \in I}$. The scheme constructed by gluing together this family is $\mathbf{A}^n_{\boldsymbol{S}}$. This construction is independent of the affine open cover chosen for $\boldsymbol{S}$ (proof omitted).

The projective space over a scheme: For any scheme $\boldsymbol{S}$ and $n > 0$ we also define an associated scheme called the n-dimensional **projective space** over $\boldsymbol{S}$, denoted $\mathbf{P}^n_{\boldsymbol{S}}$.

First, assume that $\boldsymbol{S}$ is isomorphic to Spec R for some ring R. We define $\mathbf{P}^n_{\boldsymbol{S}}$, which we also denote $\mathbf{P}^n_R$ in this case. We do this by gluing together $n+1$ copies of $\mathbf{A}^n_R = \text{Spec } R[x]$, as follows:

1. Let $R[X]$ denote the ring $R[X_0, \ldots, X_n]$.
2. For each i in $[0, n]$:
   a. Let $\alpha_i\colon R[x] \to R[X][X_i^{-1}]$ be the map given by (1) renumbering the variables x_1 through x_n as x_0 through x_n in order, skipping x_i; and (2) replacing each x_j with X_j/X_i. For example, $\alpha_1(x_1^2 + x_2) = (X_0/X_1)^2 + (X_2/X_1)$. This map is an isomorphism of $R[x]$ onto its image.
   b. Let $R_i = \alpha_i(R[x])$. Then $R_i \cong R[x]$. Let $\boldsymbol{S}_i = \text{Spec } R_i = (T_i, \boldsymbol{O}_i)$, and let $T_i = (S_i, O_i)$. Then $\boldsymbol{S}_i \cong \mathbf{A}^n_R$.

c. For each $j \neq i$ in $[0, n]$, let $U_{ij} \subseteq S_i$ be the distinguished open set (§ 4.1) that is the complement of the zero set of X_j/X_i. By definition, we have $\boldsymbol{O}_i(U_{ij}) = R_i[(X_j/X_i)^{-1}] = R_i[X_i/X_j]$ (§ 4.2).

3. For each U_{ij}, the map $X_j/X_i \mapsto X_i/X_j$ takes U_{ij} to U_{ji} and takes $\boldsymbol{O}_i(U_{ij})$ to $\boldsymbol{O}_j(U_{ji})$. Therefore these maps induce isomorphisms $\Phi_{ij}\colon \boldsymbol{S}_i|U_{ij} \to \boldsymbol{S}_j|U_{ji}$, which we may use to glue the schemes $\boldsymbol{S}_i$. Because $\boldsymbol{S}_i \cong \mathbf{A}^n_R$, this gluing induces a gluing of the $n+1$ copies of $\mathbf{A}^n_R$, which we call $\mathbf{P}^n_R$.

Now suppose that $\boldsymbol{S}$ is a scheme covered by affine schemes $\{\boldsymbol{S}_i\}_{i \in I}$. Construct each scheme $\mathbf{P}^n_{R_i}$ as described above. From the construction, and from the properties of the gluing construction, each $\mathbf{P}^n_{R_i}$ is covered by affine schemes $\{\boldsymbol{S}'_{ij}\}_{j \in J}$. Now for each pair (i, j, k) in $I \times J \times I$ use the construction given for the affine space over a scheme to glue $\boldsymbol{S}'_{ij}$ to $\boldsymbol{S}'_{kj}$. The resulting scheme is $\mathbf{P}^n_{\boldsymbol{S}}$.

6.2. Fibered Products

In this section we define a construction called the **fibered product** of two schemes $\boldsymbol{S}_1$ and $\boldsymbol{S}_2$ over a third scheme $\boldsymbol{S}$, written $\boldsymbol{S}_1 \times_{\boldsymbol{S}} \boldsymbol{S}_2$.

The tensor product as a fibered coproduct: First we review a basic property of the tensor product of modules (see *Definitions for Commutative Algebra*, § 7). Let R be a ring, let A, B, and C be R-modules, and consider the diagram shown in Figure 2. Note the following:

1. Because $r \otimes_R 1 = 1 \otimes_R r$ for all r in R, the rectangle in the lower right commutes.
2. For any module homomorphisms ϕ_A and ϕ_B such that the solid arrows commute, there exists a unique homomorphism ϕ that makes all the arrows commute. This homomorphism is given by

$$\phi(\sum_i a_i \otimes b_i) = \sum_i \phi_A(a_i) \otimes \phi_B(b_i).$$

These properties make $A \otimes_R B$ into the **fibered coproduct** or **fibered sum** of A and B over R in the category R-Mod of R modules. A fibered coproduct is also called a **pushout**.[11]

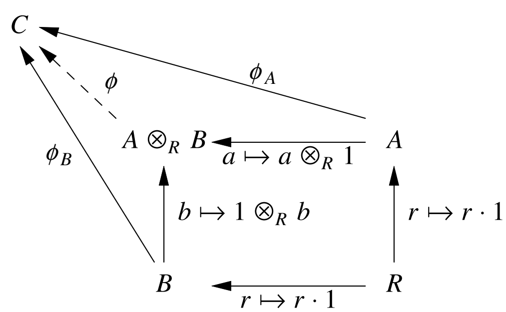

Figure 2: The tensor product as a fibered coproduct.

In the same way we can construct the fibered coproduct of R-algebras. Recall that an R-algebra is a ring A that is also an R-module, such that the map $f\colon R \to A$ given by $r \mapsto r \cdot 1_A$ is a ring homomorphism. In particular, the ring R is an R-algebra. The tensor product of R-modules naturally yields a tensor product of R-algebras. See *Definitions for Commutative Algebra*, § 9. Therefore Figure 2 is also valid in the category of R-algebras.

The fibered product of affine schemes: Now let $\boldsymbol{S}_1$, $\boldsymbol{S}_2$, and $\boldsymbol{S}$ be the affine schemes Spec A, Spec B, and Spec R. Assume there are scheme morphisms $\Phi'_1\colon \boldsymbol{S}_1 \to \boldsymbol{S}$ and $\Phi'_2\colon \boldsymbol{S}_2 \to \boldsymbol{S}$. From § 5.10, we know that the category of affine schemes is the category of commutative rings with arrows reversed. Therefore the scheme morphisms Φ'_1 and Φ'_2 correspond to ring homomorphisms $\phi'_1\colon R \to A$ and $\phi'_2\colon R \to B$, and these homomorphisms make A and B into R-modules via restriction of scalars (see *Definitions for Commutative Algebra*, § 7). Further, if we define

$$\boldsymbol{S}_1 \times_{\boldsymbol{S}} \boldsymbol{S}_2 = \text{Spec } (A \otimes_R B),$$

then we obtain the diagram shown in Figure 3, where $\boldsymbol{S}_3 = \text{Spec } C$. We obtain this diagram by applying Spec to each of the rings in Figure 2, applying the definitions, and reversing all arrows. Figure 3 has the analogous universal

[11] Compare the discussion of pullbacks and pushouts in § 12 of *Definitions for Category Theory*.

property to Figure 2, i.e., for any Φ_1 and Φ_2 that make the solid arrows commute, there is a unique Φ that makes all the arrows commute.

Figure 3 shows that $\boldsymbol{S}_1 \times_{\boldsymbol{S}} \boldsymbol{S}_2$ is the **fibered product** of $\boldsymbol{S}_1$ and $\boldsymbol{S}_2$ over $\boldsymbol{S}$ in the category of affine schemes. A fibered product is also called a **pullback**. Fibered products are useful because they allow us to define operations such as taking the product of two schemes, or taking the preimage of a morphism between two schemes, in a way that corresponds to the analogous operations in the category of sets.

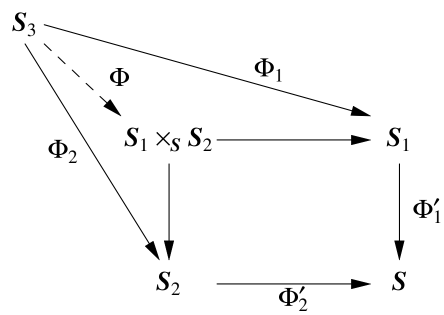

Figure 3: The fibered product of schemes.

The fibered product of general schemes: Next, consider Figure 3 in the case where $\boldsymbol{S}_1$, $\boldsymbol{S}_2$, and $\boldsymbol{S}$ are general schemes, and the morphisms $\Phi'_1 \colon \boldsymbol{S}_1 \to \boldsymbol{S}$ and $\Phi'_2 \colon \boldsymbol{S}_2 \to \boldsymbol{S}$ are given. Let $\Phi'_1 = (\phi'_1, \psi'_1)$, and $\Phi'_2 = (\phi'_2, \psi'_2)$. By definition, we can cover $\boldsymbol{S}$ with affine open subschemes $\boldsymbol{S}|_{W_k}$. Let U_k be the inverse image of W_k under ϕ'_1. Then U_k is open, so we can form $\boldsymbol{S}_1|_{U_k}$ and cover it with open affine schemes $\boldsymbol{S}_1|_{U_{ki}}$. Similarly, we can cover the inverse image V_k of W_k under ϕ'_2 with open affine schemes $\boldsymbol{S}_2|_{V_{kj}}$. The morphisms Φ'_1 and Φ'_2 induce morphisms

$$\Phi'_{1ki} \colon \boldsymbol{S}_1|_{U_{ki}} \to \boldsymbol{S}|_{W_k} \quad \text{and} \quad \Phi'_{2kj} \colon \boldsymbol{S}_2|_{V_{kj}} \to \boldsymbol{S}|_{W_k}.$$

Now for each triple (i, j, k) we can use the fibered product of affine schemes to construct the scheme

$$(\boldsymbol{S}_1 \times_{\boldsymbol{S}} \boldsymbol{S}_2)_{ijk} = \boldsymbol{S}_1|_{U_{ki}} \times_{\boldsymbol{S}|_{W_k}} \boldsymbol{S}_2|_{V_{kj}}.$$

The gluing construction for schemes (§ 6.1) then yields a unique way to glue the schemes $(\boldsymbol{S}_1 \times_{\boldsymbol{S}} \boldsymbol{S}_2)_{ijk}$ together into a single scheme $\boldsymbol{S}_1 \times_{\boldsymbol{S}} \boldsymbol{S}_2$ (proof omitted). This is the fibered product of the general schemes $\boldsymbol{S}_1$ and $\boldsymbol{S}_2$ over the general scheme $\boldsymbol{S}$.

The fiber of a morphism over a point: Let $\boldsymbol{S}_1 = ((S_1, O_1), \boldsymbol{O}_1)$ and $\boldsymbol{S}_2 = ((S_2, O_2), \boldsymbol{O}_2)$ be schemes, and let $\Phi \colon \boldsymbol{S}_1 \to \boldsymbol{S}_2$ be a morphism. Let p be a point of S_2. Let Spec R be an affine open subscheme of $\boldsymbol{S}_2$ that contains p; by the definition of a scheme, this scheme exists. Then p corresponds to a prime ideal $\mathbf{p}$ of R, and we may form the field $K_{\mathbf{p}}$ (§ 4.1). Because $K_{\mathbf{p}}$ is a field, it has a single prime ideal (the zero ideal), and so the scheme Spec $K_{\mathbf{p}}$ has a single point. We define the **fiber** of Φ over p, written $\Phi^{-1}(p)$, to be the fibered product $\boldsymbol{S}_1 \times_{\boldsymbol{S}_2} K_{\mathbf{p}}$. This definition does not depend on the choice of R (proof omitted).

This definition is motivated by the situation in the category of sets: there, given a map $\phi \colon S_1 \to S_2$, a point p in S_2, and the inclusion map $\{p\} \to S_2$, the fibered product $S_1 \times_{S_2} \{p\}$ is $\phi^{-1}(p) \times \{p\} \subseteq S_1 \times \{p\}$. See Figure 4. The maps π_i are the projection maps. Again the fibered product has the universal property that for any set S and maps ϕ_1 and ϕ_2 that make the solid arrows commute, there is a unique map ϕ that makes all the arrows commute. In this case the fibered product is the "fiber over p" in an intuitive sense: it is the Cartesian product of the preimage $\phi^{-1}(p)$ with the one-point set $\{p\}$.

Let $\Phi = (\phi, \psi)$. If $\boldsymbol{S}_1 = ((S_1, O_1), \boldsymbol{O}_1)$ and $\boldsymbol{S}_2 = ((S_2, O_2), \boldsymbol{O}_2)$ are the affine schemes Spec R_1 and Spec R_2, then we have the following (proof omitted):

1. The points of $\Phi^{-1}(\mathbf{p})$ are the prime ideals $\mathbf{q}$ of S_1 such that $\phi(\mathbf{q}) = \mathbf{p}$.
2. If $\mathbf{p}$ is a closed point,[12] then $\Phi^{-1}(p)$ is the scheme Spec I R_1, where $I \subseteq R_1$ is the ideal $\psi_{S_2}(\mathbf{p})$. Here ψ_{S_2} is the element of the family of maps $\psi = \{\psi_U\}$ that maps $R_2 = \boldsymbol{O}(S_2)$ to $R_1 = \boldsymbol{O}(\phi^{-1}(S_2)) = \boldsymbol{O}(S_1)$. See § 5.10.

[12] That is, $\mathbf{p}$ is a maximal ideal. See § 4.1.

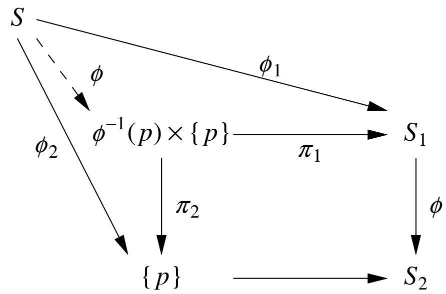

Figure 4: The fiber over a point in the category of sets.

The preimage of a closed subscheme: Again let $\boldsymbol{S}_1 = ((S_1, O_1), \boldsymbol{O}_1)$ and $\boldsymbol{S}_2 = ((S_2, O_2), \boldsymbol{O}_2)$ be schemes, and let $\Phi = (\phi, \psi): \boldsymbol{S}_1 \to \boldsymbol{S}_2$ be a morphism. Let $\boldsymbol{S}$ be a closed subscheme of $\boldsymbol{S}_2$ (§ 5.6). We define the **preimage** or **inverse image** of $\boldsymbol{S}$ under Φ, written $\Phi^{-1}(\boldsymbol{S})$, to be the fibered product $\boldsymbol{S}_1 \times_{\boldsymbol{S}_2} \boldsymbol{S}$. We claim the following (proofs omitted):

1. $\Phi^{-1}(\boldsymbol{S})$ is a closed subscheme of $\boldsymbol{S}_1$.
2. In general, the ideal sheaf $\boldsymbol{J}$ associated to $\Phi^{-1}(\boldsymbol{S})$ is $\boldsymbol{J}(U) = \psi_U(\boldsymbol{I}(U))\ \boldsymbol{O}(S_1)$, where $\boldsymbol{I}$ is the ideal sheaf associated to $\boldsymbol{S}$.
3. In the case $\boldsymbol{S}_1 = \text{Spec } R_1$ and $\boldsymbol{S}_2 = \text{Spec } R_2$, the ideal J associated to $\Phi^{-1}(\boldsymbol{S})$ is $\psi_{S_2}(I)\ R_1$, where I is the ideal associated to $\boldsymbol{S}$. Note that this statement agrees with what we said above in the case that $\boldsymbol{S}$ is a closed point.

6.3. The Category of *S*-Schemes

Fix a scheme $\boldsymbol{S}$. A **scheme over** $\boldsymbol{S}$ or $\boldsymbol{S}$**-scheme** is a pair $(\boldsymbol{S}_1, \Phi_1)$, where $\boldsymbol{S}_1$ is a scheme, and $\Phi_1: \boldsymbol{S}_1 \to \boldsymbol{S}$ is a scheme morphism. We make the set of $\boldsymbol{S}$-schemes into a category as follows:

1. The objects are the $\boldsymbol{S}$-schemes.
2. A morphism $\Phi: (\boldsymbol{S}_1, \Phi_1) \to (\boldsymbol{S}_2, \Phi_2)$ is a scheme morphism $\Phi: \boldsymbol{S}_1 \to \boldsymbol{S}_2$ such that $\Phi_2 \circ \Phi = \Phi_1$.

This construction makes $\boldsymbol{S}$ a **terminal object** in the category of $\boldsymbol{S}$-schemes, i.e., every object in the category has exactly one arrow to $\boldsymbol{S}$. As a result, the fibered product $\boldsymbol{S}_1 \times_{\boldsymbol{S}} \boldsymbol{S}_2$ is the category-theoretic product $\boldsymbol{S}_1 \times \boldsymbol{S}_2$ in the category of $\boldsymbol{S}$-schemes.[13]

There are at least two reasons why it is useful to introduce $\boldsymbol{S}$-schemes:

1. Fix a morphism $\Phi_1: \boldsymbol{S}_1 \to \boldsymbol{S}$. Then for each point p in $\boldsymbol{S}$ there is a corresponding scheme, i.e., the fiber of Φ_1 over p (§ 6.2). Therefore we can think of Φ_1 as a family of schemes parameterized by the points of $\boldsymbol{S}$.
2. In some domains (a) each scheme $\boldsymbol{S}_1$ maps to $\boldsymbol{S}$ in a natural way and (b) the morphisms $\Phi_{12}: \boldsymbol{S}_1 \to \boldsymbol{S}_2$ that we care about are the ones that are compatible with the maps $\Phi_1: \boldsymbol{S}_1 \to \boldsymbol{S}$ and $\Phi_2: \boldsymbol{S}_2 \to \boldsymbol{S}$.

Schemes over a field K**:** As an example of item 2, consider the case where $\boldsymbol{S}$ is the affine scheme Spec K, where K is a field. In this case we say that $\boldsymbol{S}_1$ is a **scheme over** K or a K**-scheme**. By the theorem in § 5.10, a morphism from $\boldsymbol{S}_1 = ((S_1, O_1), \boldsymbol{O}_1)$ to $\boldsymbol{S} = \text{Spec } K$ corresponds to a homomorphism from K to $\boldsymbol{O}_1(S_1)$, and this homomorphism makes $\boldsymbol{O}_1(S_1)$ into a K-algebra by restriction of scalars. The morphisms in the category of K-schemes are the morphisms of schemes that respect the K-algebra structure. In many domains these are the morphisms we care about. For example, when translating classical algebraic geometry to the language of scheme theory, we work in the category of schemes over an algebraically closed field K.

Schemes over Z: As another example, consider the case where $\boldsymbol{S}$ is the affine scheme Spec $\mathbf{Z}$. In this case we say that $\boldsymbol{S}_1$ is a **scheme over Z** or a **Z-scheme**. Every ring is a $\mathbf{Z}$-algebra, via the map that takes $n \cdot 1$ to the sum of 1 n times (if $n \geq 0$) or the sum of the additive inverse of 1 n times (if $n < 0$). Therefore there is a homomorphism from $\mathbf{Z}$ to $\boldsymbol{O}_1(S_1)$, which corresponds to a morphism from $\boldsymbol{S}_1$ to $\mathbf{Z}$. Thus every scheme $\boldsymbol{S}_1$ is a $\mathbf{Z}$-scheme, and the category of $\mathbf{Z}$-schemes is just the category of schemes.

[13] See *Definitions for Category Theory*, § 11.1. Compare the situation in the category of sets, where the fibered product of two sets over a point is the Cartesian product of the sets.

6.4. Separated Schemes

In the theory of manifolds, we require that the topological space $T = (S, O)$ associated with a manifold be Hausdorff, i.e., for any two distinct points a and b in S, there exist open sets U_a and U_b in O such that $a \in U_a$, $b \in U_b$, and $U_a \cap U_b = \varnothing$. T is Hausdorff if and only if the **diagonal set** $\Delta = \{(a, a): a \in S\}$ is a closed subset of $S \times S$ under the product topology induced by O.[14]

In the theory of schemes, we can't require that the topological space T associated with a scheme be Hausdorff, because the Zariski topology is not Hausdorff. However, we can make an analogous definition. Let $\boldsymbol{S}'$ be a scheme, and let $(\boldsymbol{S}, \Phi)$ be an $\boldsymbol{S}'$-scheme. Let $\boldsymbol{S} = ((S, O), \boldsymbol{O})$, let $\Phi = (\phi, \psi)$, and let $\boldsymbol{S}' = ((S', O'), \boldsymbol{O}')$.

1. Assume that $\boldsymbol{S} = \text{Spec } R$ and $\boldsymbol{S}' = \text{Spec } R'$, so that $\boldsymbol{S} \times_{\boldsymbol{S}'} \boldsymbol{S} = \text{Spec } (R \otimes_{R'} R)$ (§ 6.2). We define the **diagonal ideal** $I_\Delta(\boldsymbol{S} \times_{\boldsymbol{S}'} \boldsymbol{S}) \subseteq R \otimes_{R'} R$ to be the ideal generated by the set $\{r \otimes 1 - 1 \otimes r: r \in R\}$.
2. Now let $\boldsymbol{S}$ and $\boldsymbol{S}'$ be general schemes. We say that $\boldsymbol{S}$ is **separated over $\boldsymbol{S}'$** or is a **separated $\boldsymbol{S}'$-scheme** if there exists a closed subscheme $\Delta = ((S_\Delta, O_\Delta), \boldsymbol{O}_\Delta)$ of $\boldsymbol{S} \times_{\boldsymbol{S}'} \boldsymbol{S}$ with the following property:
   a. Let the ideal sheaf of $\boldsymbol{S}_\Delta$ be $\boldsymbol{I}_\Delta$ (§ 5.6).
   b. We require that for each pair of affine open sets $U \subseteq S$ and $U' \subseteq S'$ with $\phi(U) \subseteq U'$

$$\boldsymbol{I}_\Delta(U) = I_\Delta(\boldsymbol{S}|_U \times_{\boldsymbol{S}'|_{U'}} \boldsymbol{S}|_U).$$

We say that a scheme $\boldsymbol{S}$ is **separated** if it is separated over $\mathbf{Z}$.

Any affine scheme is separated over Spec $\mathbf{Z}$ or over any other affine scheme, because in this case Δ is the closed subscheme associated to the diagonal ideal $I_\Delta(\boldsymbol{S} \times_{\boldsymbol{S}'} \boldsymbol{S})$. In general, separated schemes are the schemes of interest in the theory. When gluing schemes together (§ 6.1) we must be careful to ensure that the result is a separated scheme.

6.5. The Spectrum of a Sheaf

In § 4 we described Spec R, the spectrum of a commutative ring R. In this section we describe a generalization of this concept called Spec $\boldsymbol{F}$, the spectrum of a quasicoherent sheaf $\boldsymbol{F}$ of $\boldsymbol{O}$-algebras (§ 5.4) on a scheme $\boldsymbol{S} = (T, \boldsymbol{O})$. Spec $\boldsymbol{F}$ is also called a **global spectrum**, because it describes the structure associated with an entire sheaf. In contrast, the spectrum of a ring describes the local structure associated with an affine open set.

Spec $\boldsymbol{O}$: First we define Spec $\boldsymbol{F}$ in the special case $\boldsymbol{F} = \boldsymbol{O}$. This will motivate the more general definition.

Let $\boldsymbol{S} = (T, \boldsymbol{O})$ be an affine scheme. In this case $\boldsymbol{O}$ is a quasicoherent sheaf of $\boldsymbol{O}$-algebras on $\boldsymbol{S}$, and $\boldsymbol{S} = \text{Spec } \boldsymbol{O}(S)$. Therefore we can define Spec $\boldsymbol{O} = \text{Spec } \boldsymbol{O}(S) = \boldsymbol{S}$.

Next, let $\boldsymbol{S} = (T, \boldsymbol{O})$ be a general scheme, and cover $\boldsymbol{S}$ with affine open subschemes $\boldsymbol{S}_i = \boldsymbol{S}|_{U_i} = (T_i, \boldsymbol{O}_i)$. In this case we have the following:

1. $\boldsymbol{O}$ is a quasicoherent sheaf of $\boldsymbol{O}$-algebras on $\boldsymbol{S}$.
2. Each of the $\boldsymbol{O}_i$ is a quasicoherent sheaf of $\boldsymbol{O}_i$-algebras on $\boldsymbol{S}_i$, so Spec $\boldsymbol{O}_i = \boldsymbol{S}_i$.
3. $\boldsymbol{O}$ is obtained by gluing together the sheaves $\boldsymbol{O}_i$ (§ 6.1).

Therefore we may again define Spec $\boldsymbol{O} = \boldsymbol{S}$.

Spec $\boldsymbol{F}$: Now let $\boldsymbol{S} = (T, \boldsymbol{O})$ be a scheme, and let $\boldsymbol{F}$ be a quasicoherent sheaf of $\boldsymbol{O}$-algebras on $\boldsymbol{S}$. We define Spec $\boldsymbol{F}$ as follows:

1. Cover $\boldsymbol{S}$ with affine open subschemes $\boldsymbol{S}_i = \boldsymbol{S}_i|_{U_i} = (T_i, \boldsymbol{O}_i)$.
2. For each $\boldsymbol{S}_i$, let $\boldsymbol{F}_i$ be the quasicoherent sheaf of $\boldsymbol{O}$-algebras $\boldsymbol{F}|_{U_i}$. Define Spec $\boldsymbol{F}_\text{i} = \text{Spec } \boldsymbol{F}_i(U)$. Notice this agrees with the definition we gave for an affine scheme in the case that $\boldsymbol{F}_i = \boldsymbol{O}_i$.
3. Use the gluing construction on schemes (§ 6.1) to glue the schemes Spec $\boldsymbol{F}_i$ into the scheme Spec $\boldsymbol{F}$. The construction is similar to the one that we gave in § 6.1 for the affine space over a scheme. Note that this definition agrees with the definition we gave in the case that $\boldsymbol{F} = \boldsymbol{O}$.

This construction is independent of the open affine cover chosen for $\boldsymbol{S}$ (proof omitted).

We make the Spec $\boldsymbol{F}$ into an $\boldsymbol{S}$-scheme (§ 6.3) as follows:

[14] See, e.g., https://planetmath.org/aspacemathnormalxishausdorffifandonlyifdeltaxisclosed.

1. For each i, $\boldsymbol{F}_i(U)$ is an $\boldsymbol{O}_i(U)$-algebra. Therefore there is a ring homomorphism $\phi_i\colon \boldsymbol{O}_i(U) \to \boldsymbol{F}_i(U)$, and therefore a corresponding scheme homomorphism $\Phi_i\colon \operatorname{Spec} \boldsymbol{F}_i(U) \to \operatorname{Spec} \boldsymbol{O}_i(U)$ (§ 5.10).
2. The homomorphisms Φ_i induce, via the gluing, a homomorphism $\Phi\colon \operatorname{Spec} \boldsymbol{F} \to \boldsymbol{S}$.

When $\boldsymbol{F} = \boldsymbol{O}$, Φ is the identity morphism.

7. The Functors h and h_S

To read this section, you should be familiar with §§ 1–8 and 13 of my paper *Definitions for Category Theory*. Throughout this section we assume that our categories have small hom sets, so that (for example) we can refer to the category **Set** instead of $\mathbf{Set}_V$.

Let C be a category, and recall the following facts:

1. For any object b of C, the contravariant functor $C(-, b)\colon C^{\text{op}} \to \mathbf{Set}$ takes each object a to $C(a, b)$ (i.e., the set of morphisms from a to b) and takes each morphism $f\colon a' \to a$ to $C(f, b)$ (i.e., the morphism from $C(a, b)$ to $C(a', b)$ defined by $h \mapsto h\ f$). See *Definitions for Category Theory*, § 6.
2. The contravariant Yoneda functor $Y'\colon C \to \mathbf{Set}^{C^{\text{op}}}$ takes each object b to the functor $Y'\ b = C(-, b)$ and takes each morphism $b \to b'$ to the natural transformation $\eta = C(-, f)$ defined by $\eta_a = C(a, f) = h \mapsto f\ h$. $C(a, f)$ is an arrow from $Y'\ b\ a = C(a, b)$ to $Y'\ b'\ a = C(a, b')$. Therefore η is a natural transformation from $Y'\ b$ to $Y'\ b'$, so it is an arrow in the category $\mathbf{Set}^{C^{\text{op}}}$. See *Definitions for Category Theory*, § 13.2.

These concepts are useful in scheme theory. First, let **Sch** denote the category of schemes. Write h to denote the contravariant Yoneda functor $Y'\colon \mathbf{Sch} \to \mathbf{Set}^{\mathbf{Sch}^{\text{op}}}$. This functor is useful because it maps schemes $\boldsymbol{S}$ to functors $h\ \boldsymbol{S}$, and one can develop geometric notions in the category of functors. For example, one can define open and closed subfunctors of a functor. One can then use the geometry of functors to define the geometry of schemes.

Next, fix a scheme $\boldsymbol{S}$, and consider the functor $h_S = h\ \boldsymbol{S}$. For any scheme $\boldsymbol{S}'$, the set $h_S\ \boldsymbol{S}'$ is the set of scheme morphisms from $\boldsymbol{S}'$ to $\boldsymbol{S}$. This set is called the set of **$\boldsymbol{S}'$-valued points** of $\boldsymbol{S}$. When $\boldsymbol{S} = \operatorname{Spec} R$, we may write $\mathbf{R}$ instead of $\boldsymbol{S}$; and similarly when $\boldsymbol{S}' = \operatorname{Spec} R'$.

The concept of $\boldsymbol{S}'$-valued points has its roots in the classical problem of solving systems of equations. For example, let $\mathbf{Z}[x] = \mathbf{Z}[x_1, \ldots, x_n]$ be the ring of formal polynomials over $\mathbf{Z}$ in n variables, and let $F = \{p_\alpha\}$ be a family of polynomials in $\mathbf{Z}[x]$. Let R be a ring. For any tuple $t = (a_1, \ldots, a_n)$ of elements in R, let $\phi_t\colon \mathbf{Z}[x] \to R$ be the map defined by substituting a_i for each x_i and collecting terms.

Suppose we want to compute the set X of tuples t such that $\phi_t(p_\alpha) = 0$ for all α. Let I be the ideal generated by the polynomials in F, and let $R' = \mathbf{Z}[x]/I$. Then ϕ_t is an element of X if and only if it defines a homomorphism from R' to R. This is true if and only if ϕ_t corresponds to a scheme morphism from $\operatorname{Spec} R$ to $\operatorname{Spec} R'$ (§ 5.10). Thus, X is in bijection with the set $h_R\ R'$.

So far we have worked in the category **Sch** of schemes. These concepts apply identically in the category $\boldsymbol{S}$-**Sch** of $\boldsymbol{S}$-schemes, for any scheme $\boldsymbol{S}$ (§ 6.3).

References

Bocchino, R. *Definitions for Category Theory.* https://rob-bocchino.net/Professional/Mathematics.html.

Bocchino, R. *Definitions for Classical Algebraic Geometry.* https://rob-bocchino.net/Professional/Mathematics.html.

Bocchino, R. *Definitions for Commutative Algebra.* https://rob-bocchino.net/Professional/Mathematics.html.

Bocchino, R. *The General Derivative.* https://rob-bocchino.net/Professional/Mathematics.html.

Eisenbud, D. and Harris, J. *The Geometry of Schemes.* Springer Verlag, 1999.

Hartshorne, R. *Algebraic Geometry.* Springer Verlag, 1977.

Index