The Inverse and Implicit Mapping Theorems

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Revised October 2023

This paper presents two important theorems in higher-dimensional calculus. The **inverse mapping theorem** says that, under the right conditions, a differentiable map between normed vector spaces has a local differentiable inverse. The **implicit mapping theorem** says that under the right conditions, if we have normed vector spaces $X = X_1 \times X_2$ and Y, subsets $U_i \subseteq X_i$, and a map $f: U_1 \times U_2 \to Y$ that is differentiable at $a = (a_1, a_2)$, then by considering pairs $x = (x_1, x_2)$ in $U = U_1 \times U_2$ such that $f(x) = f(a)$, we obtain a map $g: V \subseteq U_1 \to U_2$ (the implicit map) that takes x_1 to x_2 and that is differentiable at a_1.

This paper assumes that you are familiar with the concepts presented in my paper *The General Derivative*. It also assumes that you are familiar with Cauchy and convergent sequences in normed vector spaces, as covered in my paper *Calculus over the Complex Numbers*.

For simplicity, we assume that all vector spaces are finite-dimensional over **R** or **C**. It is straightforward to generalize these concepts to infinite-dimensional vector spaces and vector spaces over other fields; we just have to specify that all vector spaces are complete and that all linear maps are continuous.[1]

1. The Inverse Mapping Theorem

In this section we discuss the inverse mapping theorem.

1.1. Preliminary Definitions

First we collect some basic definitions that we will need to state and prove the theorem.

Open balls: Let X be a normed vector space, let a be a vector in X, and let $r > 0$ be a real number. The **open ball** centered at a with radius r, written $B(a, r)$, is the set of all vectors x in X such that $|x - a| < r$. For example:

1. An open ball $B(a, r)$ in **R** is an open interval $(a - r, a + r)$.
2. An open ball $B(a, r)$ in $\mathbf{R}^2$ is a disk of radius r centered at a that does not include its boundary.

Open sets: Let X be a normed vector space, and let U be a subset of X. We say that U is **open** if, for each vector a in U, there exists a real number $r > 0$ such that $B(a, r) \subseteq U$. For example, the set of all vectors $x = (x_1, x_2)$ in $\mathbf{R}^2$ such that $x_1 \in (-1, 1)$ and $x_2 \in (-1, 1)$ is open in $\mathbf{R}^2$. Both the empty set $\varnothing$ and the entire vector space X are open.

Open neighborhoods: Let X be a normed vector space, and let a be a vector in X. An open set U containing a is called an **open neighborhood** of a.

Complements and closed sets: Let X be a normed vector space, and let $U \subseteq X$ be a subset.

1. The **complement** of U, written U^C, is the set $X - U$, i.e., the set of all points x in X such that x is not contained in U.
2. We say that U is **closed** if its complement U^C is open. For example, for any a in X and $r > 0$, the closed ball $B_{\leq}(a, r)$ consisting of all points x in X such that $|x - a| \leq r$ is closed.

Both the empty set $\varnothing$ and the entire vector space X are closed.

Maps: Let X and Y be normed vector spaces, and let $f: U \subseteq X \to V \subseteq Y$ be a map. We say that f is **injective** if it does not map any two distinct vectors in U to the same vector in V. More formally, this means that for any two vectors a and b in U, if $f(a) = f(b)$, then $a = b$. We say that f is **surjective** if every vector in V is the image $f(a)$ of some vector a in U. When both of these conditions hold, we say that f is **bijective**.

[1] A complete normed vector space is called a **Banach space**. Every finite-dimensional normed vector space over **R** or **C** is a Banach space.

We write $f(U)$ to denote the set of all elements $f(a)$ such that a is an element of U. The condition that $f : U \to V$ is surjective is equivalent to the condition $f(U) = V$.

Inverse maps: Let X and Y be normed vector spaces, and let $f : U \subseteq X \to V \subseteq Y$ be a map.

1. An **inverse map** for f is a map $f^{-1} : V \to U$ such that $f^{-1} \circ f$ is the identity map on U and $f \circ f^{-1}$ is the identity map on V. A inverse map f^{-1} exists if and only if f is bijective. When an inverse map f^{-1} exists, we say that f is **invertible**.
2. Let $W \subseteq U$ be an open subset. The **restriction map** $f|_W : W \to V$ is the map f restricted to the domain W, i.e., the map $a \mapsto f(a)$ for all vectors a in W.
3. Let a be a vector in U. We say that f has a **local inverse** at a if there is an open neighborhood $W \subseteq U$ of a such that $f|_W$ is injective. In this case there is a map $g : W \to f(W)$ such that $g(a) = f(a)$ for all a in W, and g has an inverse g^{-1}.

Order of differentiability: Let X and Y be finite-dimensional normed vector spaces over $\mathbf{R}$ or $\mathbf{C}$, let $f : U \subseteq X \to Y$ be a map, let p be a point in U, and let $n > 0$ be a natural number. We say that f is **differentiable to order** n at p if $D^i f(p)$ exists for all $i \in [1, n]$. We say that f is **infinitely differentiable** at p if $D^i f(p)$ exists for all $i > 0$. We say that f is differentiable to order n (respectively infinitely differentiable) if it has that property at every point in its domain.

Note that if $D^n f$ exists for $n > 1$, then Df^{n-1} is continuous, because differentiability implies continuity. Accordingly, we make the following definition. If $D^n f$ exists and is continuous, then we say that f is **continuously differentiable to order** n **.** An infinitely differentiable function is continuously differentiable to all orders.

1.2. An Example

We now present a simple example from first-year calculus. Let $f : \mathbf{R} \to \mathbf{R}$ be the function $f(x) = x^2$. Then f has no local inverse at zero. Indeed, choose any open set W containing zero. Then for any positive number a that is sufficiently close to zero, both a and $-a$ are in W, and $a^2 = (-a)^2$. Therefore f is not injective when restricted to W.[2]

On the other hand, f does have a local inverse at any point $a \neq 0$. For example, let $a = 2$, and let W be the open interval $(1, 3)$. Then $f(a) = 2^2 = 4$, and $f(W)$ is the open interval $(1, 9)$. There is only one number x in W such that $f(x) = 4$, and that is $x = 2$. The other real number x such that $x^2 = 4$, namely $x = -2$, is not a member of W.

In general, f has a local inverse at any point a where f is either increasing or decreasing for all points sufficiently close to a, i.e., its derivative at a is not zero. In the case of $f(x) = x^2$, we have $Df(x) = 2x$, so $Df(a) = 0$ if and only if $a = 0$. In § 1.5, we shall see that a general map f has a local inverse at points a where its derivative $Df(a)$ is invertible as a linear map $\lambda : \mathbf{R} \to \mathbf{R}$.

Let W be an open subset of $\mathbf{R}$ that does not contain zero. From first-year calculus, we know that the local inverse $g^{-1} : f(W) \to W$ is given by $g(x) = x^{1/2}$. We also know that g^{-1} is differentiable on $f(W)$, with derivative $Dg^{-1}(y) = (1/2)\ y^{-1/2}$. Substituting $y = f(x) = x^2$, we obtain

$$Dg^{-1}(f(x)) = \frac{1}{2} \cdot \frac{1}{(x^2)^{1/2}} = \frac{1}{2x} = Df(x)^{-1}, \tag{1}$$

where $Df(x)^{-1}$ denotes the inverse of $Df(x)$ as a linear map. In § 1.5, we shall see that equation (1) is a specific case of a general rule for the derivative of a local inverse.

1.3. Preliminary Results

To prove the inverse mapping theorem, we will need the following results.

1.3.1. Contraction Maps

The proof of the inverse mapping theorem depends upon a key fact about a special kind of map from a normed vector space to itself. Let X be a normed vector space, let $U \subseteq X$ be a subset, and let $f : U \to U$ be a map. We say that f is a **contraction map** or **shrinking map** with constant c if (a) c is a real number such that $0 < c < 1$, and (b) for any vectors a and b in U, we have

[2] By convention, we write $\sqrt{a}$ or $a^{1/2}$ to denote the nonnegative square root of a. Note that the function $f(x) = x^{1/2}$ does not satisfy the definition of an inverse in a neighborhood of zero; for example, when $W = (-2, 2)$, we have $((-1)^2)^{1/2} = 1^{1/2} = 1 \neq -1$.

$$|f(a) - f(b)| \le c|a - b|. \tag{2}$$

For example, let $f\colon \mathbf{R} \to \mathbf{R}$ be the map $x \mapsto x/2$. Then f is a contraction map with constant $c = 1/2$, because for any a and b in $\mathbf{R}$ we have

$$|f(a) - f(b)| = \left|\frac{a}{2} - \frac{b}{2}\right| = \frac{1}{2}\,|a - b|,$$

so (2) holds with $c = 1/2$. Observe the following facts about this map:

1. We have $f(0) = 0/2 = 0$. Therefore point $x = 0$ is a **fixed point** of f, i.e., a point a such that $f(a) = a$.
2. For any point a, we have $f(a) = a/2$. Therefore f moves a closer to zero, unless a is already zero. Further, $\lim_{n\to\infty} f^n(a) = \lim_{n\to\infty} \dfrac{a}{2^n} = 0$, where f^n denotes $f \circ \cdots \circ f$ (n times).

We generalize these observations with the following **contraction lemma**:

Let X be a finite-dimensional normed vector space over $\mathbf{R}$ or $\mathbf{C}$, let U be a nonempty closed subset of X, and let $f\colon U \to U$ be a contraction map with constant c. Then

1. *f has a fixed point p, i.e., a point p in U such that $f(p) = p$.*
2. *The fixed point p is unique, i.e., for any fixed point q, we have $q = p$.*
3. *For any point a in U, we have $\lim_{n\to\infty} f^n(a) = p$.*

Proof: (1) Choose a point a in U. We will show that $p_a = \lim_{n\to\infty} f^n(a)$ exists and is a fixed point of f.

Let i, j, and k be positive integers with $i = j + k$. Applying (2) j times yields

$$|f^i(a) - f^j(a)| = |f^j(f^k(a)) - f^j(a)| \le c^j |f^k(a) - a|. \tag{3}$$

Further,

$$\begin{aligned}
|f^k(a) - a| = |a - f^k(a)| &= |a + \sum_{n=1}^{k-1}(-f^n(a) + f^n(a)) - f^k(a)| \\
&= |\sum_{n=0}^{k-1}(f^n(a) - f^{n+1}(a))| \\
&\le \sum_{n=0}^{k-1} |f^n(a) - f^{n+1}(a)| \\
&\le \sum_{n=0}^{k-1} c^n |a - f(a)| \\
&\le \frac{1}{1-c}\,|a - f(a)|,
\end{aligned}$$

where we have used the triangle inequality to move the norm bars inside the sum, and the last step follows from the convergence of the geometric series.[3] The last term is a constant N, independent of i, j, and k. Therefore (3) yields

$$|f^i(a) - f^j(a)| \le c^j N,$$

and by taking large enough j we can make the right-hand side arbitrarily small. Therefore the sequence $S_a = \{f^i(a)\}_{i \in \mathbf{N}}$ (where $\mathbf{N}$ denotes the natural numbers $0, 1, 2, \ldots$) is Cauchy;[4] and because X is finite-dimensional over $\mathbf{R}$ or $\mathbf{C}$ and therefore complete, S converges to an element p_a in X. It is a basic fact about closed sets in a topological space that if S is a sequence of points in a closed set $U \subseteq X$, and S converges to a point q in X, then U contains q.[5] Therefore U contains p_a.

[3] See *Calculus over the Complex Numbers*, § 4.2.

[4] See *Calculus over the Complex Numbers*, § 4.2.

[5] See, e.g., [Gaal 2009]. Here is a simple proof in the case of a normed vector space. It suffices to prove the contrapositive, i.e., if q is not contained in U, then no sequence of points in U converges to q. Because U is closed, its complement U^C is open. Therefore there exists an open

To see that p_a is a fixed point of f, consider the absolute difference

$$\begin{aligned}|f(p_a) - p_a| &= |f(p_a) - f^n(a) + f^n(a) - p_a| \quad (n > 0)\\ &\le |f(p_a) - f^n(a)| + |f^n(a) - p_a|\\ &\le c|p_a - f^{n-1}(a)| + |f^n(a) - p_a|.\end{aligned}$$

For large enough n, we can make both terms on the right arbitrarily small, so the left-hand side must be zero, i.e., $f(p_a) = p_a$.

(2) Suppose p and q are fixed points of f. Then we have

$$|p - q| = |f(p) - f(q)| \le c|p - q|.$$

If $|p - q| \ne 0$, then we can divide through by this term, yielding $1 \le c$. But $c < 1$ by assumption. Therefore $|p - q| = 0$, i.e., $p = q$.

(3) This fact follows from the proofs of (1) and (2). □

1.3.2. The Map $\lambda \mapsto \lambda^{-1}$

We will also need the fact that the map $\lambda \mapsto \lambda^{-1}$ is an infinitely differentiable map from a subset of $L(X, Y)$ to $L(Y, X)$. As usual, $L(X, Y)$ denotes the space of linear maps from X to Y.

Let X and Y be finite-dimensional normed vector spaces over $\mathbf{R}$ or $\mathbf{C}$. Let $U \subseteq L(X, Y)$ be an open set of invertible linear maps. Let $f\colon U \to L(Y, X)$ be the map $\lambda \mapsto \lambda^{-1}$. Then f is infinitely differentiable.

Proof: Fix a point λ in U. We first show that Df exists at λ. Choose a point $\lambda_1 \in L(X, Y)$ such that $\lambda + \lambda_1 \in U$, and consider the difference map

$$\Delta = f(\lambda + \lambda_1) - f(\lambda) = (\lambda + \lambda_1)^{-1} - \lambda^{-1}.$$

Fix a point $x \in X$, and let $y = (\lambda + \lambda_1)(x)$. Then

$$\Delta(y) = x - x - \lambda^{-1}(\lambda_1(x)) = -(\lambda^{-1}(\lambda_1((\lambda + \lambda_1)^{-1}(y))).$$

Therefore

$$\Delta = (\lambda + \lambda_1)^{-1} - \lambda^{-1} = -\lambda^{-1} \circ \lambda_1 \circ (\lambda + \lambda_1^{-1}),$$

i.e.,

$$(\lambda + \lambda_1)^{-1} = \lambda^{-1} - \lambda^{-1} \circ \lambda_1 \circ (\lambda + \lambda_1^{-1}). \tag{4}$$

Substituting the right-hand side of (4) for $(\lambda + \lambda_1)^{-1}$ in the right-hand side, we obtain

$$\begin{aligned}f(\lambda + \lambda_1) = (\lambda + \lambda_1)^{-1} &= \lambda^{-1} - \lambda^{-1} \circ \lambda_1 \circ (\lambda^{-1} - \lambda^{-1} \circ \lambda_1 \circ (\lambda + \lambda_1^{-1}))\\ &= f(\lambda) + g(\lambda_1) + \phi(\lambda_1),\end{aligned} \tag{5}$$

where

$$g(\lambda_1) = -\lambda^{-1} \circ \lambda_1 \circ \lambda^{-1}$$

and

$$\phi(\lambda_1) = \lambda^{-1} \circ \lambda_1 \circ \lambda^{-1} \circ \lambda_1 \circ (\lambda + \lambda_1^{-1}).$$

Then g is a composition of linear maps, so it is linear. Therefore by the definition of the derivative and (5), we have $g(\lambda_1) = Df(\lambda)(\lambda_1)$ if ϕ is $o(\lambda_1)$. But this is true because

$$|\phi(\lambda_1)| \le |\lambda^{-1}||\lambda_1||\lambda^{-1}||\lambda_1||\lambda + \lambda_1^{-1}|,$$

and dividing by $|\lambda_1|$ leaves a factor of $|\lambda_1|$ that goes to zero as λ_1 goes to zero.

ball $B(q, r)$ contained in U^C. This means that every point in U has at least distance r to q, so no sequence of points in U can get arbitrarily close to q.

Thus we have shown that $f(\lambda) = \lambda^{-1}$ has the first derivative

$$D^1 f(\lambda) = (\lambda_1 \mapsto -f(\lambda) \circ \lambda_1 \circ f(\lambda)) \tag{6}$$

everywhere on U. Now we examine the higher-order derivatives. Rewrite (6) as follows:

$$D^1 f(\lambda)(\lambda_1) = -f(\lambda) \circ (\lambda_1 \circ f(\lambda)). \tag{7}$$

The outer composition in (7) is a composition of linear maps, which is a bilinear map. Therefore we can apply the product rule (*The General Derivative*, § 7.4) to the outer composition. Doing that yields

$$D^2 f(\lambda)(\lambda_1)(\lambda_2) = -Df(\lambda)(\lambda_2) \circ (\lambda_1 \circ f(\lambda)) - f(\lambda) \circ D(\lambda_1 \circ f(\lambda))(\lambda_2). \tag{8}$$

By the rule for composition with a linear map (*The General Derivative*, § 7.6), we have

$$D^2 f(\lambda)(\lambda_1)(\lambda_2) = -Df(\lambda)(\lambda_2) \circ \lambda_1 \circ f(\lambda) - f(\lambda) \circ \lambda_1 \circ Df(\lambda)(\lambda_2).$$

$$= f(\lambda) \circ \lambda_2 \circ f(\lambda) \circ \lambda_1 \circ f(\lambda) + f(\lambda) \circ \lambda_1 \circ f(\lambda) \circ \lambda_2 \circ f(\lambda). \tag{9}$$

We can then repeat this process, generating a derivative of any desired order. □

Example: Identify $\mathbf{R}$ with $L(\mathbf{R}, \mathbf{R})$ according to the isomorphism $r \mapsto M(r)$. (Recall that $M(r)$ is the linear map "multiply by r."). Then an element λ of $L(\mathbf{R}, \mathbf{R})$ corresponds to a number r, and λ^{-1} corresponds to $1/r$. Let $f : \mathbf{R} - \{0\} \to \mathbf{R}$ be the map $(\lambda \mapsto \lambda^{-1}) = (r \mapsto 1/r)$. In this context we compose linear maps by multiplying numbers. So by (6), $Df(r)$ is the linear map $M(-1/r^2) = h \mapsto -h/r^2$. Indeed,

$$f(r+h) - f(r) - Df(r)(h) = \frac{1}{r+h} - \frac{1}{r} + \frac{h}{r^2} = \frac{h^2}{r^2(r+h)},$$

which is $o(h)$. Notice also that the formula Df agrees with the rule learned in first-year calculus for the derivative of the function $f(x) = 1/x$.

1.4. The Weak Inverse Mapping Theorem

We now state and prove a weak form of the inverse mapping theorem. This form contains some assumptions that make the proof easier, and that we will relax in § 1.5.

Let X be a finite-dimensional normed vector space over $\mathbf{R}$ or $\mathbf{C}$. Fix an open neighborhood U of 0 in X and a map $f : U \to X$ that takes 0 to 0. Assume that f is continuously differentiable to order $n > 0$, that the derivative $Df(x)$ is invertible at each point $x \in U$, and that $Df(0)$ is the identity map $I : X \to X$. Then f has a local inverse at 0, i.e., there exists an open neighborhood $W \subseteq U$ of 0 and a map $g : W \to f(W)$ such that $g = f$ on W and g has an inverse g^{-1}. Moreover, g^{-1} is continuously differentiable to order n, and at each point y in $f(W)$ we have $Dg^{-1}(y) = Df(g^{-1}(y))^{-1}$.

Proof: Let $F : U \to X$ be the mapping $x \mapsto x - f(x)$. Then $DF(0) = 0$, and DF is continuous on U, so there exists a real number $r > 0$ such that

$$x \in B_{\leq}(0, r) \Rightarrow |DF(x)| \leq \frac{1}{2}.$$

Fix such an r, and let $W = B(0, r) \cap f^{-1}(B(0, r/2))$. f is differentiable and therefore continuous. Therefore $f^{-1}(B(0, r/2))$ is open, so W is an intersection of open sets and therefore an open neighborhood of zero.

We wish to show that $f|_W$ is injective, i.e., for any y in $f(W)$ there exists a unique x_y in W such that $f(x_y) = y$. It suffices to show that for any y in $B_{\leq}(0, r/2)$, there exists a unique x_y in $B_{\leq}(0, r)$ such that $f(x_y) = y$, because in this case, for any y in $f(W)$,

1. y is in $B_{\leq}(0, r/2)$, so there is a unique x_y in $B_{\leq}(0, r)$ such that $f(x_y) = y$.
2. x_y is in W and $W \subseteq B_{\leq}(0, r)$, so if x_y is unique in $B_{\leq}(0, r)$, then it must unique in W.

Let x_1 and x_2 be any points in $B_{\leq}(0, r)$, and let $h = x_2 - x_1$. By the generalized mean value theorem (*The General Derivative*, § 7.8), we have

$$|F(x_1) - F(x_2)| = |F(x_1) - F(x_1 + h)| = \left| \int_0^1 DF(x_1 + th)(h)\ dt \right| \leq \int_0^1 |DF(x_1 + th)(h)|\ dt$$

$$\leq \int_0^1 |DF(x_1 + th))||h|\ dt \leq \int_0^1 \frac{1}{2}\ |h|\ dt = \frac{1}{2}\ |x_1 - x_2|. \tag{10}$$

In particular, setting $x_1 = x$ and $x_2 = 0$, we have

$$x \in B_{\leq}(0, r) \Rightarrow |F(x)| \leq \frac{1}{2}\,|x|. \tag{11}$$

For any point y in $B_{\leq}(0, r/2)$, define $F_y\colon B_{\leq}(0, r) \to B_{\leq}(0, r)$ as follows:

$$F_y(x) = y + F(x) = x + (y - f(x)).$$

The range in the definition of F_y is well-defined because

$$\begin{aligned}
|F_y(x)| &= |y + F(x)| \leq |y| + |F(x)| \\
&\leq \frac{r}{2} + |F(x)| \ \ \text{(because } y \in B_{\leq}(0, r/2)\text{)} \\
&\leq \frac{r}{2} + \frac{|x|}{2} \ \ \text{(by (11))} \\
&\leq \frac{r}{2} + \frac{r}{2} \ \ \text{(because } x \in B_{\leq}(0, r)\text{)} \\
&= r.
\end{aligned}$$

Further, F_y is a contraction map with constant 1/2, because for any points x_1 and x_2 in $B_{\leq}(0, r)$, we have

$$\begin{aligned}
|F_y(x_1) - F_y(x_2)| &= |F(x_1) - F(x_2)| \\
&\leq \frac{1}{2}\,|x_1 - x_2| \ \ \text{(by 10).}
\end{aligned}$$

Define

$$x_y = \lim_{n \to \infty} F_y^n(0).$$

By § 1.3.1, x_y is well-defined, is a member of $B_{\leq}(0, r)$, and is a fixed point of F_y, i.e.,

$$F_y(x_y) = x_y + (y - f(x_y)) = x_y,$$

so $f(x_y) = y$ as required. Further, by the uniqueness of the fixed point, x_y is the only point in $B_{\leq}(0, r)$ with this property.

We have established that $f|_W$ is injective, so there exists a map $g\colon W \to f(W)$ such that g equals f on W and g has an inverse g^{-1}. We now show that g^{-1} is continuous on $f(W)$. For all x in W, we have $x = f(x) + F(x)$. Therefore for all x_1 and x_2 in W, we have

$$\begin{aligned}
|x_1 - x_2| &= |f(x_1) + F(x_1) - f(x_2) - F(x_2)| \\
&= |f(x_1) - f(x_2) + F(x_1) - F(x_2)| \\
&\leq |f(x_1) - f(x_2)| + |F(x_1) - F(x_2)| \\
&\leq |f(x_1) - f(x_2)| + \frac{1}{2}\,|x_1 - x_2| \ \ \text{(by (10)).}
\end{aligned}$$

Moving the second term on the right to the left and collecting terms yields

$$|x_1 - x_2| \le 2|f(x_1) - f(x_2)|,$$

so for all y_1 and y_2 in $f(W)$, we have

$$|g^{-1}(y_1) - g^{-1}(y_2)| \le 2|y_1 - y_2|. \tag{12}$$

Inequality (12) establishes that g^{-1} is continuous.

We now show that g^{-1} is continuously differentiable. Choose elements $y \in f(W)$ and $h \in X$ such that $y + h \in f(W)$. Let $x_y = g^{-1}(y)$ and $x_{y+h} = g^{-1}(y + h)$. Then x_y and x_{y+h} both lie in $B_{\le}(0, r)$. Consider the difference function

$$\phi(h) = g^{-1}(y + h) - g^{-1}(y) - Df(x_y)^{-1}(h).$$

To show that g^{-1} is differentiable at y with derivative $Dg^{-1}(y) = Df(x_y)^{-1} = Df(g^{-1}(y))^{-1}$, we need to show that ϕ is $o(h)$, i.e., $\phi(h)/|h|$ tends to zero as h tends to zero.

Let $k = x_{y+h} - x_y$. Then $h = f(x_{y+h}) - f(x_y) = f(x_y + k) - f(x_y)$, and

$$\phi(h) = k - Df(x_y)^{-1}(f(x_y + k) - f(x_y)). \tag{13}$$

Because f is differentiable at x_y, we have

$$f(x_y + k) = f(x_y) + Df(x_y)(k) + \psi(k), \tag{14}$$

where ψ is $o(k)$. Substituting (14) into (13) and canceling terms yields

$$\phi(h) = Df(x_y)^{-1}(\psi(k)). \tag{15}$$

Further,

$$|Df(x_y)^{-1}(\psi(k))| \le |Df(x_y)^{-1}||\psi(k)|,$$

and $|Df(x_y)^{-1}|$ is independent of k, so it suffices to show that $\psi(k)$ is $o(h)$. As h tends to zero, $k = g^{-1}(y + h) - g^{-1}(y)$ tends to zero by the continuity of g^{-1}, and so $\psi(k)/|k|$ tends to zero because ψ is $o(k)$. Thus it suffices to show that $|k| \le 2|h|$ for all h in $f(W)$. But this is true because by (12), we have

$$|k| = |g^{-1}(y + h) - g^{-1}(y)| \le 2|y + h - y| = 2|h|.$$

The derivative $Dg^{-1}(y) = Df(g^{-1}(y))^{-1}$ is continuous, because it is the composition of the following continuous maps:

1. g^{-1}, which is continuous by what we proved above.
2. Df, which is continuous by hypothesis.
3. $\lambda \mapsto \lambda^{-1}$, which is differentiable and therefore continuous by § 1.3.2.

Now for the higher-order derivatives. If the order n in the statement of the theorem is 1, we are done. Otherwise, let F be the function $\lambda \mapsto \lambda^{-1}$ defined on invertible linear maps in $L(X, X)$, and write

$$Dg^{-1} = F \circ Df \circ g^{-1} = G \circ g^{-1}, \tag{16}$$

where $G = F \circ Df$. By assumption f has $n \ge 2$ continuous derivatives, and by § 1.3.2 F has infinitely many continuous derivatives. Therefore G is continuously differentiable, and we may apply the chain rule to (16), yielding the continuous derivative

$$D^2 g^{-1}(x) = (DG \circ g^{-1})(x) \circ Dg^{-1}(x). \tag{17}$$

If $n = 2$, we are done. Otherwise by the chain rule we have the continuous derivative

$$DG(x) = (DF \circ Df)(x) \circ D^2 f(x). \tag{18}$$

By applying the product rule to the outer composition in (18) and the chain rule to the inner composition in (18), analogously to what we did in § 1.3.2, we can form the continuous derivative $D^2 G(x)$. We can repeat this process $n - 2$ times, forming $n - 1$ continuous derivatives of G. Now we can apply the same procedure to (17), forming n continuous derivatives of g^{-1}. □

1.5. The Inverse Mapping Theorem

Now we state and prove the stronger form of the theorem.

> *Let X and Y be finite-dimensional normed vector spaces over $\mathbf{R}$ or $\mathbf{C}$. Fix an open subset $U \subseteq X$ and a map $f: U \to Y$. Assume that f is continuously differentiable to order $n > 0$ and that the derivative $Df(x)$ is invertible at each point $x \in U$. Then at each point $p \in U$, f has a local inverse, i.e., there exists an open neighborhood $W \subseteq U$ of p and a map $g: W \to f(W)$ such that $g = f$ on W and g has an inverse g^{-1}. Moreover, g^{-1} is continuously differentiable to order n, and at each point y in $f(W)$ we have $Dg^{-1}(y) = Df(g^{-1}(y))^{-1}$.*

Proof: First we prove the theorem in the case that $p = f(p) = 0$. Let $\lambda: X \to Y$ be the linear map $Df(p) = Df(0)$, and consider the inverse map $\lambda^{-1}: Y \to X$, which exists by assumption. Let $f_1: U \to X = \lambda^{-1} \circ f$. Then $f_1(0) = 0$, and $Df_1(0) = \lambda^{-1} \circ Df(0) = \lambda^{-1} \circ \lambda = I$. Moreover, we have

$$f = \lambda \circ f_1.$$

By § 1.4, there exists an open neighborhood $W \subseteq U$ of 0 and a map $g_1: W \to f_1(U)$ such that $g_1 = f_1$ on W, g_1 has an inverse g_1^{-1}, and for each $x \in W$ g_1^{-1} is continuously differentiable to order n at $y_1 = f_1(x)$ with $Dg_1^{-1}(y_1) = Df_1(x)^{-1}$. Therefore, there exists a map $g = \lambda \circ g_1: W \to f(W)$ such that $g = f$ on W, g has an inverse $g^{-1} = g_1^{-1} \circ \lambda^{-1}$, and g^{-1} is continuously differentiable to order n at $y = f(x)$. Moreover,

$$Dg^{-1}(y) = Dg_1^{-1}(y) \circ \lambda^{-1} = Df_1(x)^{-1} \circ \lambda^{-1}$$

and

$$Df(x) = \lambda \circ Df_1(x).$$

Therefore $Dg^{-1}(y) = Df(x)^{-1}$, as was to be shown.

Now we relax the assumption $p = f(p) = 0$. Let $h_1: X \to X$ be the map $x \mapsto x + p$, let $h_2: Y \to Y$ be the map $y \mapsto y - f(p)$, and consider the map $f_2 = h_2 \circ f \circ h_1: h_1^{-1}(U) \to h_2(f(U))$. Then f_2 maps zero to zero. Moreover, we have

$$f = h_2^{-1} \circ f_2 \circ h_1^{-1}.$$

By the result just shown, there exists an open neighborhood $W_1 \subseteq h_1^{-1}(U)$ of $h_1^{-1}(p) = 0$ and a map $g_2: W_1 \to f_2(W_1)$ such that $g_2 = f_2$ on W_1, g_2 has an inverse g_2^{-1}, and for all $x \in W_1$ g_2^{-1} is continuously differentiable to order n at $y = f_2(x)$ with $Dg_2^{-1}(y) = Df_2(x)^{-1}$. Therefore there exists an open neighborhood $W = h_1(W_1) \subseteq U$ of p and a map $g = h_2^{-1} \circ g_2 \circ h_1^{-1}: W \to f(W)$ such that $g = f$ on W, g has an inverse $g^{-1} = h_1 \circ g_2^{-1} \circ h_2$, and g^{-1} is continuously differentiable to order n at $y = f(x)$. Moreover, the derivatives of h_1 and h_2 and their inverses map every vector to the identity map I, so

$$Dg^{-1}(y) = Dg_2^{-1}(h_2(y)) = Df_2(g_2^{-1}(h_2(y)))^{-1} = Df_2(h_1^{-1}(x))^{-1}$$

and

$$Df(x) = Df_2(h_1^{-1}(x)).$$

Therefore $Dg^{-1}(y) = Df(x)^{-1}$, as was to be shown. □

2. The Implicit Mapping Theorem

In this section we discuss the implicit mapping theorem.

2.1. An Example

Again we start with a simple example from first-year calculus. Let $f: \mathbf{R}^2 \to \mathbf{R}$ be the function

$$f(x) = f(x_1, x_2) = x_1^2 + x_2^2, \tag{1}$$

and consider the equation $f(x) = 1$. The set of points x satisfying equation (1) is the unit circle centered at the origin in $\mathbf{R}^2$. Observe the following:

1. $D_2 f(x_1, x_2) = 2x_2$.

2. Let $p = (0, 1)$. Then $D_2 f(p) = 2 \neq 0$. Let U_1 be a small neighborhood of 0, say $U_1 = B(0, 1/2)$, and let U_2 be a small neighborhood of 1, say $U_2 = B(1, 1/2)$. Consider the set S of points $x = (x_1, x_2)$ in $U = U_1 \times U_2$ such that $f(x) = 1$. Then the relation $g(x_1) = x_2$ for all (x_1, x_2) in S defines a function $g\colon U_1 \to U_2$. This function is given by $g(x_1) = \sqrt{1 - x_1^2}$, and it is differentiable with derivative

$$Dg(x_1) = \frac{1}{2}\,(1 - x_1^2)^{-1/2}\,(-2x_1) = \frac{-x_1}{\sqrt{1 - x_1^2}}\,.$$

3. Let $q = (1, 0)$. Then $D_2 f(q) = 0$. Let U_1 be a small neighborhood of 1, say $U_1 = B(1, 1/2)$, and let U_2 be a small neighborhood of 0, say $U_2 = B(0, 1/2)$. Consider the set S of points $x = (x_1, x_2)$ in $U = U_1 \times U_2$ such that $f(x) = 1$. Then the relation $g(x_1) = x_2$ does not yield a well-defined function $g\colon U_1 \to U_2$, because for each $x_1 \neq 1$ in S_1, there are two numbers x_2 such that $f(x_1, x_2) = 1$, namely $\sqrt{1 - x_1^2}$ and $-\sqrt{1 - x_1^2}$.

The map g in item 2 is called an **implicit map**. In general, for a map $f\colon X_1 \times X_2 \to Y$, an implicit map $g(x_1) = x_2$ exists near points p where f is differentiable and $Df_2(p)$ is invertible as a linear map.

2.2. The Weak Implicit Mapping Theorem

As before, we first state and prove a weak form of the theorem.

Let X_1 and X_2 be finite-dimensional normed vector spaces over $\mathbf{R}$ or $\mathbf{C}$, let $U_1 \subseteq X_1$ and $U_2 \subseteq X_2$ be open sets, let $U = U_1 \times U_2$, and let $f\colon U \to X_2$ be a map. Assume that f is continuously differentiable to order $n > 0$ and that the derivative $Df(x)$ is invertible at each point $x \in U$. Let $a = (a_1, a_2)$ be a point in U, and assume that $D_2 f(a) = I$. Let $b = f(a)$. Then there exists an open neighborhood W_1 of a_1 in U_1 and a map $g\colon W_1 \to U_2$ such that $g(a_1) = a_2$, $f(x_1, g(x_1)) = b$ for all x_1 in W_1, and g is continuously differentiable to order n.

Proof: Let $\phi\colon U \to U_1 \times X_2$ be the map $(x_1, x_2) \mapsto (x_1, f(x_1, x_2))$. Taking the derivative of ϕ yields

$$D\phi(a) = \begin{bmatrix} I_{X_1} & 0 \\ D_1 f(a) & D_2 f(a) \end{bmatrix} = \begin{bmatrix} I_{X_1} & 0 \\ D_1 f(a) & I_{X_2} \end{bmatrix}.$$

As a linear map, $D\phi(a)$ has an inverse

$$D\phi(a)^{-1} = \begin{bmatrix} I_{X_1} & 0 \\ -D_1 f(a) & I_{X_2} \end{bmatrix}.$$

Therefore by § 1.5 there exists an open neighborhood $W \subseteq U$ of a and a map $\chi\colon W \to \phi(W)$ such that $\chi = \phi$ on W, χ has an inverse $\chi^{-1} = \psi$, and ψ is continuously differentiable to order n on $\chi(W)$.

Let ψ_1 and ψ_2 be the coordinate maps of ψ, i.e., for all $x = (x_1, x_2)$ in $\chi(W)$, let

$$\psi(x_1, x_2) = (\psi_1(x_1, x_2), \psi_2(x_1, x_2)).$$

Then $\psi_1(x_1, x_2) = x_1$, and ψ_2 is continuously differentiable to order n. Let W_1 be the set of elements x_1 such that $(x_1, x_2) \in W$ for some $x_2 \in X_2$. Then W_1 is an open neighborhood of a_1 in U_1. Define the mapping $g\colon W_1 \to U_2$ by

$$g(x_1) = \psi_2(x_1, b).$$

Then g is continuously differentiable to order n. Further, for all x_1 in W_1, we have

$$(x_1, f(x_1, g(x_1))) = \phi(x_1, g(x_1)) = \phi(\psi_1(x_1, b), \psi_2(x_1, b))$$

$$= \phi(\psi(x_1, b)) = (x_1, b).$$

Therefore $f(x_1, g(x_1)) = b$, as required. □

2.3. The Implicit Mapping Theorem

Now we state and prove the stronger form of the theorem.

*Let X_1, X_2, and Y be finite-dimensional normed vector spaces over **R** or **C**, let $U_1 \subseteq X_1$ and $U_2 \subseteq X_2$ be open sets, let $U = U_1 \times U_2$, and let $f: U \to Y$ be a map. Assume that f is continuously differentiable to order $n > 0$ and that the derivative $Df(x)$ is invertible at each point $x \in U$. Let $a = (a_1, a_2)$ be a point in U, and let $b = f(a)$. Then there exists an open neighborhood W_1 of a_1 in U_1 and a map $g: W_1 \to U_2$ such that $g(a_1) = a_2$, $f(x_1, g(x_1)) = b$ for all x_1 in W_1, and g is continuously differentiable to order n. Moreover, there exists a real number $r > 0$ such that the values $g(x)$ are uniquely determined for all $x \in B(a_1, r)$.*

Proof: Let $\lambda: X_2 \to Y$ be the linear map $D_2 f(a)$, and consider the inverse map $\lambda^{-1}: Y \to X_2$. Let

$$f^*: U \to X_2 = \lambda^{-1} \circ f.$$

Then $D_2 f^*(a) = I$. Let $b^* = f^*(a)$. By § 2.2 there exists an open neighborhood W_1 of a_1 in U_1 and a map $g: W_1 \to U_2$ such that $g(a_1) = a_2$, g is continuously differentiable to order n, and

$$f^*(x_1, g(x_1)) = b^* \tag{2}$$

for all x_1 in W_1. Applying λ to both sides of (2), we see that $f(x_1, g(x_1)) = b$, as required.

As to the uniqueness of g on $B(a_1, r)$, it suffices to show the uniqueness result for the map f^*. Fix an open neighborhood W_1 of a_1 and a map g as guaranteed in § 2.2, and choose $r > 0$ such that $B(a_1, r) \subseteq W_1$. Let V_1 be an open neighborhood of of a_1, and let $h: V_1 \to U_2$ be a continuous map such that $h(a_1) = a_2$ and $f^*(x_1, h(x_1)) = b$ for all $x_1 \in V_1$. Let $S = V_1 \cap B(a_1, r)$. It suffices to show that g and h attain the same values on S.

Let ϕ be as in § 2.2. For all $x_1 \in S$, we have

$$\phi(x_1, h(x_1)) = (x_1, f^*(x_1, h(x_1))) = (x_1, b)$$

and

$$\phi(x_1, g(x_1)) = (x_1, f^*(x_1, g(x_1))) = (x_1, b)$$

and therefore

$$\phi(x_1, h(x_1)) = \phi(x_1, g(x_1)). \tag{3}$$

Let W and W_1 be as defined in § 2.2. ϕ is invertible on W, so for all x_1 such that $(x_1, h(x_1)) \in W$, (3) implies $h(x_1) = g(x_1)$. Moreover, $(x_1, g(x_1)) \in W$ for all $x_1 \in S \subseteq B(a_1, r) \subseteq W_1$. Further, $g(a_1) = a_2 = h(a_1)$, and g and h are continuous. Therefore there exists an open set $T \subseteq S$ containing a_1 such that $(x_1, h(x_1)) \in W$ for all $x_1 \in T$, and so $h = g$ on T. For example, let B_2 be an open ball around a_2 contained in W. By the continuity of the map $x_1 \mapsto (x_1, h(x_1))$, we may choose an open ball B_1 around a_1 contained in S such that $(x_1, h(x_1)) \in B_2$ for all $x_1 \in B_1$. Then we may let $T = B_1$.

We now show that S itself is such a set T. Choose $x_1 \in S$, and let $v = x_1 - a_1$. Let Z be the set of real numbers t such that $0 \le t \le 1$ and $g(a_1 + tv) = h(a_1 + tv)$. Then Z is not empty, so it has a least upper bound. Let s be a real number in Z. By definition, $g(a_1 + sv) = h(a_1 + sv)$. If $s < 1$, then all the conditions of the present theorem are satisfied at $a_1 + sv$, so we can reassert all the arguments made thus far to establish that g and h are equal in a neighborhood of $a_1 + sv$. Therefore s is not the least upper bound if $s < 1$. Hence the least upper bound is 1, i.e., $Z = S$. □

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