Calculus over the Complex Numbers

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This paper sketches the theory of differentiation and integration over the complex numbers. It can serve as a complement to a standard textbook on complex analysis, e.g., [Lang 1999]. The textbooks provide more detail; this document hits the highlights and shows how they fit together. It also shows how complex analysis builds naturally on real analysis.

This document assumes that you are familiar with the material covered in my papers *The General Derivative* and *Integration in Real Vector Spaces*. It develops calculus over the complex numbers as a special case of calculus in general vector spaces with real norms. I find this approach more satisfactory than the typical approach, which develops complex analysis as an ad-hoc extension of calculus in two real variables.

1. The Complex Numbers

There is no real number r such that $r^2 = -1$. Intuitively, the complex numbers **C** are the smallest extension of the real numbers **R** that contains such a number and that preserves the operations of addition, subtraction, multiplication, and division. In this section we make this idea precise.

1.1. The Algebraic Definition

We begin with an algebraic definition of the complex numbers. This definition is the most general because it doesn't depend on a choice of real coordinates. For computation, real coordinates are often useful; we discuss these in the next section.

First we define the algebraic concept of a **field**. A field F is an algebraic construct that generalizes the real numbers. It is a set together with two binary operations $o \colon F \times F \to F$:

1. **Addition**, written $a + b$. Addition is associative and commutative. It has an additive identity, i.e., an element 0 or zero such that for all elements a in F, $a + 0 = 0 + a = a$. For every element a in F, there is an additive inverse of a, i.e., a unique element $-a$ in F such that $a + (-a) = (-a) + a = 0$.
2. **Multiplication**, written ab or $a \cdot b$. Multiplication is associative and commutative. It has a multiplicative identity, i.e., an element 1 or one such that for all elements a in F, $a \cdot 1 = 1 \cdot a = a$. For every element a in F except zero, there is a multiplicative inverse of a, i.e., a unique element a^{-1} or $1/a$ in F such that $a \cdot a^{-1} = a^{-1} \cdot a = 1$. Multiplication distributes over addition, i.e., for every elements a, b, and c in F, we have $a(b + c) = ab + ac$.

It is clear that the real numbers **R** are a field.

With this definition in hand, we may define the complex numbers **C** as follows:

C1. **C** is a field.

C2. **C** contains **R** as a **subfield**. That is, (a) every element of **R** is an element of **C**; and (b) the operations, identities, and inverses of **R** as a subset of **C** are identical to those of **R**.

C3. **C** contains an element i such that $i^2 = -1$.[1]

C4. **C** is minimal in the sense that no subset of **C**, except **C** itself, satisfies properties **C1** through **C3**.

This definition specifies **C** algebraically as an **extension field** of **R** (i.e., a field that has **R** as a subfield). For more on the properties of extension fields, see my paper *Definitions for Commutative Algebra*.

[1] The letter i stands for "imaginary." There is a long tradition of calling the square root of negative one an "imaginary number," even though it is no more imaginary than any other mathematical abstraction.

We now show how **C** may be regarded as a vector space. In *The General Derivative*, we defined a vector space as a set of vectors satisfying the rules of vector addition and scalar multiplication by elements of **R**. We now generalize this definition. We say that a vector space as defined in *The General Derivative* is a **real vector space** or a **vector space over the real numbers**. Analogously, we can define a vector space V over any field F. It is a set of vectors satisfying the rules of vector addition and scalar multiplication by elements of F. The definition is identical to the one given in *The General Derivative*, after replacing **R** with F as the field of scalars.

Any field F is a vector space over itself. In particular, **C** is a vector space over itself, just as **R** is a vector space over itself. **C** is also a vector space over **R**.

In the more general setting, we define a normed vector space as follows. Let F be a field of scalars s. An **absolute value** on F is a function that assigns to each scalar s a real number $|s|$ and that has the same properties as the absolute value of a real number, namely:

1. For any scalar s, $|s| \geq 0$, and $|s| = 0$ if and only if $s = 0$.
2. For any two scalars s_1 and s_2, $|s_1 s_2| = |s_1||s_2|$.
3. For any two scalars s_1 and s_2, $|s_1 + s_2| \leq |s_1| + |s_2|$.

Let $|s|$ be an absolute value on F, and let V be a vector space over F. A **norm** on V is a function that assigns to each vector v in V a real number $|v|$, and that has the properties of the norm stated in § 2 of *The General Derivative*, namely:

1. For any vector v, $|v| \geq 0$, and $|v| = 0$ if and only if $v = 0$.
2. For any scalar s and vector v, $|sv| = |s||v|$.
3. For any two vectors v_1 and v_2, $|v_1 + v_2| \leq |v_1| + |v_2|$.

Notice that the norm is always a real number, even when V is a vector space over a field $F \neq$ **R**. As stated in *The General Derivative*, **R** is a normed vector space over itself, with the norm of r given by the absolute value $|r|$. In § 1.3, we shall see that **C** is a normed vector space as well.

1.2. Real Coordinates

It is often useful to represent complex numbers as pairs of real numbers. There are two standard ways to do this: rectangular coordinates and polar coordinates.

1.2.1. Rectangular Coordinates

To develop rectangular coordinates for complex numbers, we observe the following:

1. By property **C3**, i is a complex number.
2. By properties **C1** and **C2**, for any real numbers x and y, $x + iy$ is a complex number.

Thus it is natural to define the **rectangular coordinate map** $R\colon \mathbf{R}^2 \to \mathbf{C}$ given by $(x, y) \mapsto z = x + iy$. Note that if we restrict the scalar multiplication of **C** to real numbers we get a real vector space. Further, R is a linear map to this real vector space, because

$$R(r(x, y)) = R(rx, ry) = rx + iry = r(x + iy) = r\ R(x, y).$$

We now show that **C** is equal to $R(\mathbf{R}^2)$, i.e., the set of all $x + iy$ such that x and y are members of **R**. In fact it suffices to show that $R(\mathbf{R}^2)$ is a field; then the other properties are clear.

We make $R(\mathbf{R}^2)$ into a field as follows:

1. We define addition componentwise, i.e., $(x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2)$. The additive identity is $0 = 0 + i\ 0$, and the additive inverse of $x + iy$ is $-x + i(-y) = -x + (-iy)$. The last equality is justified because $iy + i(-y) = i(y + -y) = i\ 0 = 0$. As usual, we write $-x + (-iy)$ as $-x - iy$.
2. We define multiplication by standard polynomial multiplication, and by collecting terms. That is,

$$\begin{aligned}(x_1 + iy_1)(x_2 + iy_2) &= x_1 x_2 + iy_1\ x_2 + x_1\ iy_2 + i^2 y_1 y_2 \\ &= x_1 x_2 + i(y_1 x_2 + x_1 y_2) - y_1 y_2\end{aligned}$$

$$= (x_1 x_2 - y_1 y_2) + i(y_1 x_2 + x_1 y_2).$$

The multiplicative identity is $1 = 1 + i\,0$. To form the inverse z^{-1} of $z = x + iy$, we define the **complex conjugate** $\bar{z} = x - iy$. Then for any complex number z, $z\bar{z} = (x + iy)(x - iy) = x^2 + y^2$, so if $z \neq 0$, then the complex number $\bar{z}/(x^2 + y^2)$ is a multiplicative inverse for z.

Every complex number z has a unique representation in rectangular coordinates as $x + iy$. Therefore the rectangular coordinate map R has an inverse map $R^{-1}\colon \mathbf{C} \to \mathbf{R}^2$. This map is given by

$$z \mapsto \left(\frac{z + \bar{z}}{2}, \frac{z - \bar{z}}{2i} \right).$$

You can see this by plugging in $x + iy$ for z and $x - iy$ for $\bar{z}$ and simplifying the right-hand side to (x, y). Given a complex number $z = x + iy$, we say that $x = (z + \bar{z})/2$ is the **real part** of z and $y = (z - \bar{z})/2i$ is the **imaginary part** of z.

Because R is a bijection (it covers all of $\mathbf{C}$ and is one-to-one), we may identify the set of complex numbers $z = x + iy$ with the set of ordered pairs (x, y) in the real Cartesian plane. Under this identification,

- The complex numbers $\mathbf{C}$ are called the **complex plane**.
- The x axis is called the **real axis**.
- The y axis is called the **imaginary axis**.

A complex number $x = x + 0\,i$ is called (as in real analysis) a **real number**. A complex number $iy = 0 + iy$ is called an **imaginary number** or **pure imaginary number**.

1.2.2. Polar Coordinates

To develop polar coordinates for complex numbers, we do the following:

1. Observe that any ordered pair (x, y) in the real Cartesian plane may be written $(r \cos\theta, r \sin\theta)$, where $r = \sqrt{x^2 + y^2}$, and θ is (a) an arbitrary value (if $r = 0$) or (b) an angle in radians from the x axis to the line segment connecting the origin $(0, 0)$ to (x, y), in the counterclockwise direction.
2. Conclude that any complex number $x + iy$ has a representation $r \cos\theta + ir \sin\theta = r(\cos\theta + i \sin\theta)$, where r and θ are real numbers.
3. Define the function $e^{i\theta}\colon \mathbf{R} \to \mathbf{C}$ as $e^{i\theta} = \cos\theta + i \sin\theta$. Because $e^x\colon \mathbf{R} \to \mathbf{R}$ already denotes the real exponential function, this notation must be justified. The justification is that e^x and $e^{i\theta}$ are special cases of a function $e^z\colon \mathbf{C} \to \mathbf{C}$, with $z = x$ (i.e., z a real number) and $z = i\theta$ (i.e., z a pure imaginary number), respectively. We will define e^z in § 4.4 below.

It is easy to use the trigonometric addition formulas to establish that $e^{i\theta_1} e^{i\theta_2} = e^{i(\theta_1 + \theta_2)}$. Multiplying a complex number z by $e^{i\theta}$ displaces z by the angle θ around the origin. Positive angles correspond to counterclockwise displacement, and negative angles correspond to clockwise displacement.

With these observations and definitions in hand, we can define the **polar coordinate map** $P\colon \mathbf{R}^2 \to \mathbf{C}$ given by $(r, \theta) \mapsto z = re^{i\theta} = r \cos\theta + ir \sin\theta = R(r \cos\theta, r \sin\theta)$, where R is the rectangular coordinate map. Notice that P does not have a well-defined inverse, because (1) when $z = 0$, the value of θ is not specified at all; and (2) when $z \neq 0$, the value of r is specified only up to its sign, and the value of θ is specified only up to a multiple of 2π.

For a complex number $z = re^{i\theta}$ expressed in polar coordinates, we define the complex conjugate $\bar{z} = re^{-i\theta}$. Transforming z to rectangular coordinates, we see that $z = r \cos\theta + ir \sin\theta$ and $\bar{z} = r \cos\theta - ir \sin\theta$, because changing the sign of θ leaves $\cos\theta$ unchanged and reverses the sign of $\sin\theta$. Therefore $\bar{z}$ has the same value when z is expressed in rectangular or polar coordinates.

1.3. The Absolute Value

Fix a complex number z. We define the **absolute value** of z, written $|z|$, as follows:

1. If z is expressed in rectangular coordinates $z = x + iy$, with x and y in $\mathbf{R}$, then $|z| = \sqrt{z\bar{z}} = \sqrt{x^2 + y^2}$. Note that $|z| = |R(z)|$, where R is the real coordinate map (§ 1.2), and the vertical bars on the right represent the Euclidean norm on $\mathbf{R}^2$.

2. If z is expressed in polar coordinates $z = re^{i\theta}$, with r and θ in $\mathbf{R}$, then $z = |r|$, where $|r|$ denotes the absolute value of the real number r.

The two definitions are equivalent, because if $z = re^{i\theta}$, then according to the first definition,

$$\begin{aligned} |z| &= |r\cos\theta + i\, r\sin\theta| \\ &= \sqrt{(r\cos\theta)^2 + (r\sin\theta)^2} \\ &= \sqrt{r^2(\cos^2\theta + \sin^2\theta)} \\ &= \sqrt{r^2\cdot 1} = \sqrt{r^2} = |r|. \end{aligned}$$

Notice that when $z = x$ is a real number, the absolute value of x as a member of $\mathbf{C}$ agrees with the absolute value of x as a member of $\mathbf{R}$. Thus the complex absolute value extends the real absolute value in a natural way.

It is easy to show that the absolute value $|z|$ has the properties required for a norm on the vector space $\mathbf{C}$ (§ 1.1). Indeed, properties 1 and 3 follow directly from the representation of $R(z)$ of a complex number z in real coordinates, and the analogous properties of the norm in $\mathbf{R}^2$. As to property 2, for any two complex numbers $z_1 = r_1e^{i\theta_1}$ and $z_2 = r_2e^{i\theta_2}$, we have

$$|z_1z_2| = |r_1e^{i\theta_1}r_2e^{i\theta_2}| = |r_1r_2e^{i(\theta_1+\theta_2)}| = |r_1r_2| = |z_1||z_2|.$$

Therefore, the complex absolute value makes $\mathbf{C}$ into a normed vector space over itself.

1.4. The Topology of the Complex Plane

The complex plane has a natural structure as a topological space in two real dimensions. In this section we briefly investigate this structure.

Given a set S, a **topology** on S is a set O of subsets of S, called the **open sets** of S, satisfying certain axioms. The pair (S, O) is called a **topological space**. For more information on topological spaces, see § 23 of my paper *Definitions for Commutative Algebra*.

For any $n > 0$, the real vector space $\mathbf{R}^n$ has the following standard topology, called the **Euclidean topology**:

1. Fix a point $p = (x_1, \ldots, x_n)$ in $\mathbf{R}^n$ and a real number $r > 0$. The **open ball** $B(p, r)$ is the set of all points q in $\mathbf{R}^n$ such that $|p - q| < r$, where $|p|$ denotes the Euclidean norm $|(x_1, \ldots, x_n)| = \sqrt{x_1^2 + \cdots + x_n^2}$. In $\mathbf{R}$, the Euclidean norm of a real number r is $\sqrt{x^2} = |x|$, and an open ball is an open interval $(p - r, p + r)$. In $\mathbf{R}^2$, an open ball is a disc.
2. A set U of points in $\mathbf{R}^n$ is open if, for every point p in U, there exists a positive real number r such that $B(p, r)$ is contained in U.

For example, the set $U = \{p : |p| < 1\}$ is open in $\mathbf{R}^2$, while $V = \{p : |p| \le 1\}$ is not.

Fix a topological space (S, O) and a subset T of S.

1. The **complement** of T, written $S - T$, is the set of all points p in S such that p is not an element of T.
2. T is **closed** if its complement is open.

With reference to the example above,

1. V is closed.
2. The complement of U, i.e., $\{p : |p| \ge 1\}$, is closed.

The complex plane $\mathbf{C}$ inherits the Euclidean topology via its identification with the real plane $\mathbf{R}^2$.

1. Fix a complex number a and a real number $r > 0$. The open ball $B(a, r)$ is the set of all points z in $\mathbf{C}$ such that $|z - a| < r$.
2. A set U of points in $\mathbf{C}$ is open if, for every point a in U, there exists a positive real number r such that $B(a, r)$ is contained in U.

2. Complex Differentiation

In *The General Derivative*, we developed the theory of differentiation for maps $f: X \to Y$, where X and Y are finite-dimensional normed vector spaces over $\mathbf{R}$. This theory carries over identically to the case where X or Y are finite-dimensional normed vector spaces over $\mathbf{C}$. So in a very strong sense, we already have all the theory we need for complex differentiation. We just have to work out the details of applying the theory to particular cases. We do that in this section.

2.1. Complex-Valued Functions

In this section we discuss the derivative of a **complex-valued function**, i.e., a function that attains its values in $\mathbf{C}$.

A **complex-valued function of one complex variable** is a function $f: U \to \mathbf{C}$, where U is a subset of $\mathbf{C}$.[2] For short, we will call this kind of function a **complex function**. Fix a complex function f. From *The General Derivative*, we know that the derivative Df exists at a point z in U if and only if there exists a linear map $Df(z): \mathbf{C} \to \mathbf{C}$ such that for all complex numbers h with $|h|$ sufficiently small,

$$f(z+h) = f(z) + Df(z)(h) + \phi(h),$$

where ϕ is $o(h)$, i.e., $\lim\limits_{h \to 0} \dfrac{\phi(h)}{|h|} = 0$. Let $V \subseteq U$ be the set of points z where $Df(z)$ is defined. Then Df is a function from V to $L(\mathbf{C}, \mathbf{C})$, the vector space of linear maps from $\mathbf{C}$ to $\mathbf{C}$.

A linear map $\lambda: \mathbf{C} \to \mathbf{C}$ corresponds to multiplication by a complex number, just as a linear map $\lambda: \mathbf{R} \to \mathbf{R}$ corresponds to multiplication by a real number. As in the case of a single real variable, we refer to the number associated with the linear map $Df(z)$ as $f'(z)$ or $\dfrac{df}{dz}$, and we write

$$Df(z) = M(f'(z)) = h \mapsto f'(z)h,$$

where

$$f'(z) = \lim_{h \to 0} \frac{f(z+h) - f(z)}{h}.$$

When a function $f: \mathbf{C} \to \mathbf{C}$ is differentiable at a point z, we say that f is **holomorphic** at z. When f is holomorphic at every point z in a subset U of $\mathbf{C}$, we say that f is holomorphic on U.

A **complex-valued function of a real vector space** is a function $f: U \to \mathbf{C}$, where U is a subset of a real vector space X, for example $\mathbf{R}$ or $\mathbf{R}^2$. In this case we define the derivative of f as described in *The General Derivative*. All the theory described there goes through when we set $Y = \mathbf{C}$ in the derivative of a function $f: X \to Y$. We may also write $f = R \circ F$, where R is the rectangular coordinate map (§ 1.2.1), and $F = R^{-1} \circ f$. Then f is differentiable if and only F is a differentiable map from U to $\mathbf{R}^2$.

When $X = \mathbf{R}$, we say that f is a **complex-valued function of one real variable**. In this case everything stated above for a complex function holds, after replacing the complex number z by the real number t, the complex number h by the real number h, and "holomorphic" by "differentiable."

2.2. Properties of the Derivative

All of the properties of the derivative stated in § 7 of *The General Derivative* hold for complex-valued functions. As an example, let us prove that $\dfrac{d}{d\theta}\, e^{i\theta} = ie^{i\theta}$. Notice that $e^{i\theta}: \mathbf{R} \to \mathbf{C}$ is a complex-valued function of one real variable (§ 2.1). We have

$$\frac{d}{d\theta}\, e^{i\theta} = \frac{d}{d\theta}\,(\cos\theta + i \sin\theta) \quad \text{(definition of } e^{i\theta}\text{)}$$

$$= \frac{d}{d\theta} \cos\theta + \frac{d}{d\theta}\, i \sin\theta \quad \text{(sum rule)}$$

[2] In *Integration in Real Vector Spaces*, we wrote $f: V \to W$ to mean a partial map between vector spaces V and W, defined on some subset $U \subseteq V$. This way of writing maps simplifies the notation, but it loses precision. From now on we will be more precise and write $f: U \to W$ to mean a map from $U \subseteq V$ to W, defined everywhere on U.

$$= \frac{d}{d\theta} \cos \theta + i \frac{d}{d\theta} \sin \theta \quad \text{(composition with a linear map)}$$

$$= -\sin \theta + i \cos \theta \quad \text{(derivative of sin and cos)}$$

$$= i \, (i \sin \theta + \cos \theta) \quad \text{(definition of } i\text{)}$$

$$= ie^{i\theta} \quad \text{(definition of } e^{i\theta}\text{)}.$$

As another example, we can use the product rule and induction to prove that $\frac{d}{dz} z^n = nz^{n-1}$, where z is a complex number, $n > 0$ is an integer, and $z^n : \mathbf{C} \to \mathbf{C}$ is the function $z \mapsto z \cdots z$ (n times). Try this as an exercise.

The proof of the quotient rule goes through for complex-valued functions in the same way as for real-valued functions. Using the quotient rule, we can extend the result in the previous paragraph to all integers n, where $z^0 = 1$ and $z^{-n} = 1/z^n$ for $n > 0$.

2.3. Complex One Forms

We extend the notation of one forms to complex-valued functions. In *Integration in Real Vector Spaces*, we said that a one form is a map $\omega : U \subseteq \mathbf{R}^n \to L(\mathbf{R}^n, \mathbf{R})$. We now define a **complex one form** (one form for short when the context is clear) to be a map $\omega : U \subseteq X \to L(X, \mathbf{C})$, where X is a real vector space or $\mathbf{C}$.

Fix a complex function f. In this case, $X = \mathbf{C}$. As before, in the notation of complex one forms,

1. We write df instead of Df.
2. We write dz to denote the linear map $h \mapsto h$.
3. We write $f(z)\ dz$ or $f\ dz$ to denote the map $z \mapsto (h \mapsto f(z)h)$.

When f is holomorphic on U, we say that $f\ dz$ is holomorphic on U. When f is holomorphic at z, item 3 lets us write $df = f'\ dz$, or equivalently $df = \frac{df}{dz}\ dz$. In the second form, the two occurrences of dz appear to "cancel." This apparent "canceling" is a useful mnemonic, but one must not take it too seriously, because there are no well-defined rules for multiplying and dividing the symbols df and dz. To think rigorously about these concepts, one must go back to the definition of the derivative as a linear map.

Now fix a complex-valued function f of a real vector space. In this case, X is a real vector space, usually $\mathbf{R}$ or $\mathbf{R}^2$. The notation for a complex one form is the same as for a real one form with the corresponding structure. The only difference is that the functions appearing in the one form are complex-valued. For example, with $X = \mathbf{R}^2$, we may write $\omega = f_x\ dx + f_y\ dy$, where f_x and f_y are complex-valued functions of $\mathbf{R}^2$. This technique is useful for expressing a complex one form in real coordinates; we take up this idea further in § 3.3.

The concepts of a closed and exact one form are the same as in the real case. A one form ω is **closed** if $d\omega = 0$. It is **exact** if there exists a complex-valued function f such that $df = \omega$. For example, $f_x\ dx + f_y\ dy$ is an exact one form on a subset U of $\mathbf{R}^2$ if and only if there exists a function $f : U \to \mathbf{C}$ such that $D_x f = \frac{\partial f}{\partial x} = f_x$ and $D_y f = \frac{\partial f}{\partial y} = f_y$.

2.4. Real Vector Fields

In some applications it is useful to apply the calculus of maps from $\mathbf{R}^2$ to $\mathbf{R}^2$ to functions from $\mathbf{C}$ to $\mathbf{C}$. In this section we briefly explore this idea.

Fix a subset U of $\mathbf{C}$ and a holomorphic function $f : U \to \mathbf{C}$. Let $R : \mathbf{R}^2 \to \mathbf{C}$ be the rectangular coordinate map (§ 1.2.1), and let $V = R^{-1}(U)$, i.e., the set of all points $p = (x, y)$ in $\mathbf{R}^2$ such that $R(p) = x + iy$ lies in U. Let $F : V \to \mathbf{R}^2 = R^{-1} \circ f \circ R$. F is called the **real vector field** associated with f. It converts the function $f : U \subseteq \mathbf{C} \to \mathbf{C}$ into a corresponding function $F : V \subseteq \mathbf{R}^2 \to \mathbf{R}^2$.

Using the chain rule to take the derivative of $f \circ R$, we find

$$D(f \circ R)(p) = Df(R(p)) \circ dR(p) = M(f'(R(p))) \circ M_{22}(D_x R(p), D_y R(p)).$$

Here we use the notation from *The General Derivative* that specifies how to interpret vectors as linear maps. $M(a)$

denotes multiplication by the real number a, and $M_{22}(a, b)$ denotes the dot product with the vector (a, b). Because $R(p) = x + iy$, we have $D_x R(p) = 1$ and $D_y R(p) = i$. Therefore

$$D(f \circ R)(p) = M((f' \circ R)(p)) \circ M_{22}(1, i) = M_{22}((f' \circ R)(p), i\,(f' \circ R)(p)). \tag{1}$$

Notice that $R \circ F = R \circ R^{-1} \circ f \circ R = f \circ R$. Let (F_x, F_y) be the coordinate functions of F. Using the chain rule to take the derivative of $R \circ F$, we find

$$D(R \circ F)(p) = DR(F(p)) \circ dF(p) = [\ 1\ i\ ] \begin{bmatrix} D_x F_x(p) & D_y F_x(p) \\ D_x F_y(p) & D_y F_y(p) \end{bmatrix} \tag{2}$$

$$= M_{22}(D_x F_x(p) + i D_x F_y(p), D_y F_x(p) + i D_y F_y(p)).$$

Because $R \circ F = f \circ R$, the right-hand sides of (1) and (2) must be equal. Therefore

$$f' \circ R = D_x F_x + i D_x F_y = D_y F_y - i D_y F_x. \tag{3}$$

Comparing the real and imaginary parts on the left- and right-hand sides of (3), we find

$$D_x F_x = D_y F_y \qquad D_x F_y = -D_y F_x. \tag{4}$$

Equations (4) are called the **Cauchy-Riemann equations** associated with the real vector field F of a holomorphic function f.

3. Complex Integration

Now we turn our attention to complex integration. Integration over the complex numbers builds naturally on integration in real vector spaces. However, there are some surprising differences between the two theories, due to the differing structure of the complex numbers.

3.1. Paths

First we define the concept of a path. We also define the related concepts of homotopic paths, pathwise connected sets, and simply connected sets. We will need these concepts to carry out complex integration. Throughout this section, X denotes the real plane $\mathbf{R}^2$ or the complex plane $\mathbf{C}$.

The definition of a path: A **path** is a differentiable map $\sigma\colon s \to X$, where $s = [a, b]$ is an interval of the real line. This is the same definition that we used in *Integration in Real Vector Spaces*. When $X = \mathbf{C}$, "differentiable" means differentiable as a complex-valued function (§ 2.1). We call the ordered pair of points $E = (\sigma(a), \sigma(b))$ the **endpoints** of the path σ. We call $\sigma(a)$ the **initial point** of σ, and we call $\sigma(b)$ the **terminal point** of σ.

A path $\sigma\colon [a, b] \to U$ is **closed** if $\sigma(a) = \sigma(b)$. For example, the path $\sigma\colon [0, 2\pi] \to \mathbf{C}$ given by $\sigma(t) = e^{it}$ is closed, because $\sigma(0) = \sigma(2\pi) = 1$.

Let U be a subset of X. A **path in** U is a path $\sigma\colon [a, b] \to U$, i.e., a path $\sigma\colon [a, b] \to X$ such that for all t in $[a, b]$, $\sigma(t)$ is an element of U.

Homotopic paths: Fix a set $U \subseteq X$, an interval $s = [a, b]$, and a pair $E = (p, q)$ of points in U. Let $S_{s,E}$ be the set of paths $\sigma\colon [a, b] \to U$ with endpoints E. Fix paths σ_0 and σ_1 in $S_{s,E}$. A **homotopy** from σ_0 to σ_1 in U is a map $H\colon [0, 1] \to S_{s,E}$ such that $H(0) = \sigma_0$, $H(1) = \sigma_1$, and $(u, v) \mapsto H(u)(v)$ is a continuous map from $[0, 1] \times [a, b]$ to U. A homotopy in U defines a continuous transformation from σ_0 to σ_1 of paths in U, such that each path in the transformation has endpoints E. If there is a homotopy H from σ_0 to σ_1 in U, then there is a homotopy $u \mapsto H(1 - u)$ from σ_1 to σ_0 in U. In this case we say that the paths σ_0 and σ_1 are **homotopic** in U.

Pathwise connected sets: Let U be a subset of X. we say that U is **pathwise connected** if, for any two points p and q in U, there exists a path $\sigma\colon [a, b] \to U$ such that $\sigma(a) = p$ and $\sigma(b) = q$. For example, the open ball of radius 1 about the origin in $\mathbf{C}$ is pathwise connected. The union of the open balls of radius 1 about 2 and -2 is not pathwise connected; for example, there is no path from 2 to -2 contained in the union.

Simply connected sets: Let U be a pathwise connected subset of X. We say that U is **simply connected** if any closed path in U is homotopic to a point in U. Intuitively, U has no holes, so a closed path never winds around a hole and can therefore be continuously transformed into a point while keeping the endpoints fixed. As an example, the complex plane $\mathbf{C}$ is simply connected, while $\mathbf{C}$ with the origin deleted is not. When we delete the origin, there are paths that cannot be continuously transformed to a point while keeping the endpoints fixed. One such path starts at 1 and winds once counterclockwise around the unit circle centered on zero. The endpoints of the path are $(1, 1)$.[3] The simply connected closed subsets of $\mathbf{R}$ are the intervals $[a, b]$.

3.2. One Forms of One Real Variable

Next we show how to integrate a complex one form of one real variable, i.e., a form $f\ dt$, where $f: U \subseteq \mathbf{R} \to \mathbf{C}$ is a complex-valued function of one real variable. As we shall see in the next section, complex integration generally proceeds by pulling back to one of these forms and integrating it over a real interval $[a, b]$.

We know how to integrate $f\ dt$ where f is a function from $\mathbf{R}$ to $\mathbf{R}^2$; see, e.g., § 4 of *Integration in Real Vector Spaces*. Recall that we integrate each coordinate separately, i.e., we let f_1 and f_2 be the real-valued functions such that $f(t) = (f_1(t), f_2(t))$, and we write

$$\int_a^b f\ dt = (\int_a^b f_1\ dt, \int_a^b f_2\ dt).$$

We extend this integration to complex-valued functions f as follows. Let $f(t): [a, b] \to \mathbf{C}$ be a function defined and continuous on the real interval $[a, b]$. We define

$$\int_a^b f(t)\ dt = R\left[\int_a^b (R^{-1} \circ f)(t)\ dt\right], \tag{1}$$

where R is the rectangular coordinate map (§ 1.2.1). The integral on the right-hand side of equation (1) is well-defined, because it is the integral of the continuous function $R^{-1} \circ f: [a, b] \to \mathbf{R}^2$. Equation (1) says that to integrate a complex-valued function $f(t)$, we let f_1 and f_2 be the real-valued functions such that $f(t) = f_1(t) + i\ f_2(t)$, and we write

$$\int_a^b f\ dt = \int_a^b f_1\ dt + i \int_a^b f_2\ dt.$$

Note that $f_1 = (f + \bar{f})/2$, and $f_2 = (f - \bar{f})/2i$, where $\bar{f}_i$ is the function $t \mapsto \overline{f_i(t)}$.

Because R and R^{-1} are linear maps, we have the standard linearity property of the integral, i.e.,

$$\int_a^b (cf + dg)\ dt = c \int_a^b f\ dt + d \int_a^b g\ dt.$$

Here a and b are real numbers, and c and d are complex numbers.

The fundamental theorem of calculus holds, because we have

$$\begin{aligned} D(x \mapsto \int_a^x f(t)\ dt) &= D(x \mapsto R\left[\int_a^x (R^{-1} \circ f)(t)\ dt\right]) \\ &= D(R \circ (x \mapsto \int_a^x (R^{-1} \circ f)(t)\ dt)) \\ &= R \circ D(x \mapsto \int_a^x (R^{-1} \circ f)(t)\ dt) \ \text{ (linearity of } R) \\ &= R \circ (x \mapsto R^{-1} \circ f(x)) \ \text{ (fundamental theorem of calculus)} \\ &= f(x). \end{aligned}$$

[3] Note that if we don't fix the endpoints, then any path can be continuously transformed to a point, regardless of holes in U. For example, we can continuously retract the terminal point along the path to the initial point. By fixing the endpoints, we disallow this retraction.

Because $|R(p)| = |p|$ and $|R^{-1}(z)| = |z|$, we have the following estimate for the absolute value of the integral:

$$\left|\int_a^b f(t)\ dt\right| = \left|R\left[\int_a^b (R^{-1}\circ f)(t)\ dt\right]\right|$$

$$= \left|\int_a^b (R^{-1}\circ f)(t)\ dt\right|$$

$$\leq \int_a^b \left|(R^{-1}\circ f)(t)\right|\ dt \quad \text{(property of the real integral)}$$

$$= \int_a^b |f(t)|\ dt.$$

3.3. General One Forms

Paths: We use the techniques discussed in *Integration in Real Vector Spaces* to integrate a general one form over a path. The only difference is that here our one forms are complex-valued.

Let X be $\mathbf{R}^2$ or $\mathbf{C}$. Fix a complex one form $\omega: X \to L(X, \mathbf{C})$ and a path $\sigma: [a, b] \to X$. We define the **pullback** $\sigma^*\omega$ of ω with respect to the path σ to be

$$(\sigma^*\omega)(t) = \omega(\sigma(t)) \circ d\sigma(t).$$

We define the integral of ω over the path σ to be

$$\int_\sigma \omega = \int_a^b \sigma^*\omega. \tag{2}$$

$\sigma^*\omega$ is a map from $[a, b]$ to $L(\mathbf{R}, \mathbf{C})$, so it has the form $f\ dt$, for $f: [a, b] \to \mathbf{C}$. By § 3.2, the right-hand integral in (2) is well-defined whenever $\sigma^*\omega$ is exact, i.e., there exists $F: [a, b] \to \mathbf{C}$ such that $dF = \sigma^*\omega$. In this case, the integral in (2) evaluates to $F(b) - F(a)$. Further, when ω is exact, i.e., there exists $F: X \to \mathbf{C}$ such that $dF = \omega$, then the integral in (1) evaluates to $F(\sigma(b)) - F(\sigma(a))$, and the integral is independent of the path. This is Stokes' theorem for path integrals, which we proved in § 1 of *Integration in Real Vector Spaces*; the same proof goes through in the complex case.

When $X = \mathbf{C}$, we have $\omega = f\ dz$ and $\sigma^*\omega = f(\sigma(t))\sigma'(t)\ dt$, so

$$\int_\sigma f\ dz = \int_a^b f(\sigma(t))\sigma'(t)\ dt.$$

We will treat the case of $X = \mathbf{R}^2$ below, when we discuss real coordinates.

As an example of complex path integration, see § 1 of *Integration in Real Vector Spaces*. There we showed that the integral of the complex one form $\frac{1}{z}\ dz: \mathbf{C} - \{0\} \to L(\mathbf{C}, \mathbf{C})$ over the path $\sigma: [0, 2\pi] \to \mathbf{C}$ given by $\sigma(t) = e^{it}$ evaluates to $2\pi i$. Here is the computation again, in the more general case of $\sigma: [a, b] \to \mathbf{C} = t \mapsto e^{it}$:

$$\int_\sigma \frac{1}{z}\ dz = \int_a^b \frac{1}{\sigma(t)}\ \sigma'(t)\ dt = \int_a^b \frac{1}{e^{it}}\ ie^{it}\ dt = \int_a^b i\ dt = i(b-a).$$

Thus integrating $1/z$ over the path σ computes i times the angular displacement from $\sigma(a)$ to $\sigma(b)$. This computation is a special case of integration using polar coordinates, which we will discuss in more detail in § 3.5 below.

Notice that the integral depends on the path σ, not just the endpoints of σ: for example, a path that starts at a point p and winds back to p once counterclockwise around the unit circle has an integral of $2\pi i$; whereas a path with the same endpoints that winds twice counterclockwise around the unit circle has an integral of $4\pi i$.

Chains: As in *Integration in Real Vector Spaces*, we define a **chain** γ to be an integer-weighted formal sum of paths. For example, $\gamma = 2\sigma_1 - \sigma_2$. We define an integral over a chain to be the weighted sum of the integrals over the paths. For example,

$$\int_{2\sigma_1\omega-\sigma_2} \omega = 2\int_{\sigma_1} \omega - \int_{\sigma_2} \omega.$$

A chain is **closed** if it is a sum of closed paths.

3.4. Exact One Forms

We now turn to the question of when a one form is exact. First we review the situation for a real one form $f(x)\ dx$. If f is continuous on an interval $s = [a, b]$, then it is integrable on s, and

$$F(x) = \int_a^x f(t)\ dt$$

is defined on s. Therefore $f\ dx$ is exact on s, because by the fundamental theorem of calculus, we have $dF = f\ dx$. So we find that if f is continuous on an interval $s = [a, b]$, then $f\ dx$ is exact on s. Further, a differentiable function is continuous. Therefore if f is differentiable on s, then $f\ dx$ is exact on s.

The analogous situation for a complex one form $f(z)\ dz$ is that f is holomorphic on a simply connected subset U of **C** (§ 3.1). One can do the following:

1. Show that f is holomorphic on an open ball $B \subseteq \mathbf{C}$, then $f\ dz$ is exact on B. The proof proceeds by integrating along the boundaries of rectangles.
2. Use (1) to show that if f is holomorphic on an open subset U of **C**, then for any pair of homotopic paths (§ 3.1) σ_0 and σ_1 in U, we have

   $$\int_{\sigma_0} f\ dz = \int_{\sigma_1} f\ dz.$$

   This result is called **Cauchy's integral theorem**.
3. Use (2) to show that for any points a and z in a simply connected open subset U of **C**, and for a holomorphic function f on U, the integral of $f(\zeta)\ d\zeta$ along any path in U from a to z yields the same value. Therefore the following function is well-defined, where σ is any path from a to z:

   $$F(z) = \int_a^z f(\zeta)\ d\zeta = \int_\sigma f(\zeta)\ d\zeta.$$

4. Use (1) and (3) to show that $dF(z) = f(z)\ dz$ on U, so that $f(z)\ dz$ is exact on U.

See, e.g., [Lang 1999] for the complete proofs.

In summary, we have the following results, for a function f that is holomorphic on a simply connected open subset U of **C**:

1. The one form $f\ dz$ is exact on U, i.e., there exists a holomorphic function F on U such that $df = F\ dz$.
2. For any points a and b in **C**, the integral of $f\ dz$ over any path in U from a to b has the same value $F(b) - F(a)$.
3. The integral of $f\ dz$ over any closed path in U is zero.

These results are analogous to the results stated in *Integration in Real Vector Spaces* for a real one form $f(x)\ dx$, where U is a simply connected set in **R** (i.e., an interval).

Note that a one form $f\ dz$ may be holomorphic everywhere on an open set $U \subseteq \mathbf{C}$, but if U is not simply connected, then $f\ dz$ may not be exact on U. For example, in § 3.3, we saw that $1/z$ is holomorphic everywhere on the open set $\mathbf{C} - \{0\}$, but the integral of $1/z\ dz$ around the unit circle is not zero, and the integral depends on the path. Therefore

$1/z\ dz$ is not exact on U. It is exact on a simply connected set that does not intersect the origin. This result has no direct analog in **R**, because every pathwise connected set in **R** is also simply connected.

Note also that a one form $f\ dz$ may be holomorphic and exact on an open set U that is not simply connected. For example, the one form $1/z^2\ dz = z^{-2}\ dz$ is holomorphic and exact on $\mathbf{C} - \{0\}$, because $d(-z^{-1}) = z^{-2}\ dz$. In fact, $z^n\ dz$ is exact on $\mathbf{C} - \{0\}$ for all values of n except -1, because in all those cases we have the antiderivative rule $\int z^n\ dz = \frac{z^{n+1}}{n+1}$. In this sense $z^{-1}\ dz$ is special among all the one forms $z^n\ dz$ for integers n: it is the only one that fails to be exact on $\mathbf{C} - \{0\}$.

3.5. Real Coordinates

It is often useful to integrate a complex one form with respect to real coordinates, i.e., coordinates $p = (x, y)$ in $\mathbf{R}^2$. Let $\mu\colon \mathbf{R}^2 \to \mathbf{C}$ be a differentiable function, which we will call a **coordinate map**. Let $\sigma\colon [a, b] \to \mathbf{R}^2$ be a path in $\mathbf{R}^2$ (i.e., a differentiable function). Then the composite function $\mu \circ \sigma\colon [a, b] \to \mathbf{C}$ is a path, and we may use it to integrate a complex one form $\omega(z) = f\ dz$ as described in § 3.3. From § 3.3 of *Integration in Real Vector Spaces*, we know that

$$\int_{\mu \circ \sigma} \omega = \int_{\sigma} \mu^* \omega \tag{3}$$

when ω is a k-form; we can also give an easy proof in the case where ω is a one form $f(z)\ dz$, as follows:

$$\int_{\mu \circ \sigma} \omega = \int_a^b (\mu \circ \sigma)^* \omega \quad \text{(definition of path integration)}$$

$$= \int_a^b \omega((\mu \circ \sigma)(t)) \circ d(\mu \circ \sigma)(t) \quad \text{(definition of } (\mu \circ \sigma)^* \text{ for a one form)} \tag{4}$$

$$= \int_a^b \omega((\mu \circ \sigma)(t)) \circ d\mu(\sigma(t)) \circ d\sigma(t) \quad \text{(chain rule)}$$

$$= \int_a^b \omega(\mu(\sigma(t))) \circ d\mu(\sigma(t)) \circ d\sigma(t) \quad \text{(definition of composition)}$$

$$= \int_{\sigma} \omega(\mu(p)) \circ d\mu(p) \quad \text{(definition of path integration)} \tag{5}$$

$$= \int_{\sigma} \mu^* \omega \quad \text{(definition of } \mu^*\text{)}$$

Equation (3) says that to integrate over a composite path $\mu \circ \sigma$, we may use the coordinate map $\mu\colon \mathbf{R}^2 \to \mathbf{C}$ to pull back to $\mathbf{R}^2$ and then integrate over $\sigma\colon [a, b] \to \mathbf{R}^2$.

Rectangular coordinates: Let $\mu = R$, the rectangular coordinate map of § 1.2.1. We have $\sigma(t) = (x(t), y(t))$, $R(x, y) = x + iy$, and $(R \circ \sigma)(t) = x(t) + iy(t)$. Because $dR(x, y) = D_x R\ dx + D_y R\ dy = dx + i\ dy$, equation (5) says that

$$\int_{R \circ \sigma} f\ dz = \int_{\sigma} f(x + iy)(dx + i\ dy). \tag{6}$$

Equation (6) is the justification for the familiar formula $dz = dx + i\ dy$.[4] On the other hand, because $d(R \circ \sigma)(t) = (x'(t) + iy'(t))\ dt$, equation (4) says that

[4] This formula is a useful mnemonic. However, it can be misleading: the replacement of dz by $dx + i\ dy$ in an integral is justified not by algebraic identity, but by the rules of path integration, as explained in the text.

$$\int_{R\circ\sigma} f\ dz = \int_a^b f(x(t)+iy(t))(x'(t)+y'(t))\ dt. \tag{7}$$

Equation (7) is the one we use for computation. We can also go from (6) to (7) by applying the definition of integration over the path σ. Notice also that the following two operations give the same result:

1. Using the chain rule to compute

$$d(R\circ\sigma)(t) = dR(\sigma(x(t),y(t)))\circ d\sigma(t) = (dx+i\ dy)\circ(x'(t)\ dt, y'(t)\ dt) = (x'(t)+iy'(t))\ dt.$$

2. Using the sum rule to compute $d(x(t)+iy(t)) = (x'(t)+iy'(t))\ dt$.

More generally, one can use the chain rule to prove the sum rule.

Polar coordinates: Let μ = P, the polar coordinate map of § 1.2.2. We have $\sigma(t) = (r(t),\theta(t))$, $P(r,\theta) = re^{i\theta}$, and $(P\circ\sigma)(t) = r(t)e^{i\theta(t)}$. Because $dP(r,\theta) = D_rP\ dr + D_\theta P\ d\theta = e^{i\theta}\ dr + ire^{i\theta}\ d\theta$, equation (5) says that

$$\int_{P\circ\sigma} f\ dz = \int_\sigma f(re^{i\theta})(e^{i\theta}\ dr + ire^{i\theta}\ d\theta). \tag{8}$$

Equation (8) is the justification for the formula $dz = e^{i\theta}\ dr + ire^{i\theta}\ d\theta$.[5] On the other hand, because $d(P\circ\sigma)(t) = (r'(t)e^{i\theta(t)} + ir(t)e^{i\theta(t)}\theta'(t))\ dt$, equation (4) says that

$$\int_{P\circ\sigma} f\ dz = \int_a^b f(r(t)e^{i\theta(t)})(r'(t)e^{i\theta(t)} + ir(t)e^{i\theta(t)}\theta'(t))\ dt. \tag{9}$$

Equation (9) is the one we use for computation. We can also go from (8) to (9) by applying the definition of integration over the path σ. Notice also that the following two operations give the same result:

1. Using the chain rule to compute

$$\begin{aligned} d(P\circ\sigma)(t) &= dP(\sigma(x(t),y(t)))\circ d\sigma(t) = (e^{i\theta(t)}\ dr + ir(t)e^{i\theta(t)}\ d\theta)\circ(r'(t)\ dt, \theta'(t)\ dt) \\ &= (r'(t)e^{i\theta(t)} + ir(t)e^{i\theta(t)}\theta'(t))\ dt. \end{aligned}$$

2. Using the product rule to compute $d(r(t)e^{i\theta(t)}) = (r'(t)e^{i\theta(t)} + ir(t)e^{i\theta(t)}\theta'(t))\ dt$.

More generally, one can use the chain rule to prove the product rule.

Translated coordinates: It is useful to define coordinate maps centered away from the origin. We define the following, for a point c in **C**:

1. $R_c(x,y) = x+iy+c$. R_c is the **rectangular coordinate map centered at** c.
2. $P_c(r,\theta) = re^{i\theta}+c$. P_c is the **polar coordinate map centered at** c.

Translation does not change the derivative. Therefore

1. When integrating over a path $\sigma\colon [a,b]\to \mathbf{R}^2 = t\mapsto (x(t),y(t))$, we have

$$\begin{aligned} \int_{R_c\circ\sigma} f\ dz &= \int_\sigma f(x+iy+c)(dx+i\ dy) \\ &= \int_a^b f(x(t)+iy(t)+c)(x'(t)+y'(t))\ dt. \end{aligned} \tag{10}$$

2. When integrating over a path $\sigma\colon [a,b]\to \mathbf{R}^2 = t\mapsto (r(t),\theta(t))$, we have

$$\int_{P_c\circ\sigma} f\ dz = \int_\sigma f(re^{i\theta}+c)(e^{i\theta}\ dr + ire^{i\theta}\ d\theta)$$

[5] This formula can be misleading, for the reasons stated in the previous note.

$$= \int_a^b f(r(t)e^{i\theta(t)} + c)(r'(t)e^{i\theta(t)} + ir(t)e^{i\theta(t)}\theta'(t))\, dt. \tag{11}$$

Complex paths: It is useful to start with a complex path $\sigma\colon [a,b] \to \mathbf{C}$ and construct either or both of the following:

1. A path $\sigma_{R_c}\colon [a,b] \to \mathbf{R}^2$ such that $\sigma = R_c \circ \sigma_{R_c}$.
2. A path $\sigma_{P_c}\colon [a,b] \to \mathbf{R}^2$ such that $\sigma = P_c \circ \sigma_{P_c}$.

That way we can use rectangular coordinates to integrate over $R_c \circ \sigma_{R_c}$ or polar coordinates to integrate over $P_c \circ \sigma_{P_c}$.

In the case of rectangular coordinates, this construction is straightforward. The coordinate map R_c has an inverse R_c^{-1} given by

$$z \mapsto \left(\frac{(z-c) + \overline{(z-c)}}{2}, \frac{(z-c) - \overline{(z-c)}}{2i} \right).$$

Therefore we can let $\sigma_{R_c} = R_c^{-1} \circ \sigma$. Then it is clear that $R_c \circ \sigma_{R_c} = R_c \circ R_c^{-1} \circ \sigma = \sigma$, as desired.

In the case of polar coordinates, the situation is more complicated, because the coordinate map P_c does not have a well-defined global inverse. In particular, for any point $z = re^{i\theta} + c$ in the image of a path σ, there are many choices for the angle θ. Therefore, we make the following construction:

1. Fix the angle of the initial point of σ in a canonical way.
2. Partition σ into a sequence of sub-paths σ_i. Make each sub-path short enough that, for any point z in σ_i, there is an obvious choice for the angular displacement around c from the initial point of s_i to z.
3. Argue inductively that steps 1 and 2 provide a unique angle for each point in σ.

For example, consider the path $\sigma\colon [0,2] \to \mathbf{C}$ that starts at the point 1 on the real axis and winds once counterclockwise around the unit circle. Choose a small nonnegative number ε (e.g., choose $\varepsilon = 0.1$), and partition the interval $[0,2]$ into the sequence of sub-intervals $[0, 1-\varepsilon]$, $[1-\varepsilon, 1+\varepsilon]$, and $[1+\varepsilon, 2]$. The obvious choice for $\theta(0)$ is zero. For each interval, once we have assigned an angle to the initial point, there is an obvious choice for how the angle increases as we move through the interval. Under this choice, $\theta(1-\varepsilon)$ is a little less than π, $\theta(1)$ is π, $\theta(1+\varepsilon)$ is a little more than π, and $\theta(2)$ is 2π.

We now formalize this construction. For any point $z \neq 0$ in $\mathbf{C}$, define the **angles at** z, written A_z, to be the set of all angles θ such that $z/|z| = e^{i\theta}$. Define the **canonical angle at** z, written θ_z, to be the smallest nonnegative angle in A_z. Observe the following:

1. $0 \le \theta_z < 2\pi$.
2. A_z is the set of all elements $\theta_z + 2\pi n$, where n is an integer.

As usual, positive angles represent counterclockwise displacement, and negative angles represent clockwise displacement.

Let $\sigma\colon [a,b] \to \mathbf{C}$ be a complex path that does not pass through c, i.e., such that $\sigma(t) \neq c$ for all t in $[a,b]$. We want to construct $\sigma_{P_c}(t) = (r(t), \theta(t))$ such that

$$\sigma(t) = (P_c \circ \sigma_{P_c})(t) = r(t)e^{i\theta(t)} + c = r(t)(\cos\theta(t) + i \sin\theta(t)) + c.$$

Let $\sigma_c\colon [a,b] \to \mathbf{C} = t \mapsto \sigma(t) - c$. Then σ_c does not pass through the origin. Partition the interval $s = [a,b]$ into a sequence of n sub-intervals $\{s_i = [a_i, b_i]\}$ for $1 \le i \le n$ such that the image of each s_i under σ_c is contained in an open ball B_i that does not intersect the origin. See, e.g., [Lang 1999], III, § 4 for the formal argument that we can do this. Fix an open ball B_i and real numbers t_0 and t in s_i with $t > t_0$. Assume that $\theta(t_0)$ is a fixed member of $A_{\sigma_c(t_0)}$. Since B_i does not intersect the origin, there is a unique angle $-\pi < \delta < \pi$ such that $\theta(t_0) + \delta$ is a member of $A_{\sigma_c}(t)$. Let $\tau(t) = \sigma_c(t)/|\sigma_c(t)|$. This function is well-defined, because by assumption $\sigma_c(t) \neq 0$ on s. Also, we have

$$\delta = -i \int_{\tau(t_0)}^{\tau(t)} \frac{1}{z}\, dz.$$

The function $1/z$ is holomorphic on the simply connected set B_i, so by Cauchy's integral theorem (§ 3.4), we can integrate along any path from $\tau(t_0)$ to $\tau(t)$ in B_i and get the same result. By construction all the points $\tau(t)$ have norm one, so they all lie on the unit circle, i.e., $\tau(t) = e^{i\theta(t)}$ for some $\theta(t)$. By the computation we did in § 3.3, the integral along the unit circle $e^{i\theta}$ evaluates to i times the angular displacement δ from $\tau(t_0)$ to $\tau(t)$. Define

$$\theta(t) = \theta(t_0) + \delta.$$

This construction uniquely determines $\theta(t)$ for all t in s_i with $t > t_0$.

Now construct $r(t)$ and $\theta(t)$ as follows:

1. $r(t) = |\sigma_c(t)|$ for all t in $[a, b]$.
2. $\theta(a) = \theta_{\sigma_c(a)}$.

By the argument in the previous paragraph, $\theta(t)$ is determined for all t such that $\sigma_c(t)$ lies in B_1. Further, because the B_i are open, each B_i overlaps with B_{i+1} on the smallest values of t in s_{i+1}. Therefore by induction $\theta(t)$ is determined for all t such that $\theta(t)$ lies in any B_i, i.e., for all t in $[a, b]$.

From the definition, it is clear that $\sigma = P_c \circ \sigma_{P_c}$. Because σ is a path, $\sigma_c(t)$ is differentiable on s, and $r(t) \neq 0$ for t in s. The norm function $z \mapsto |z|$ or $x + iy \mapsto \sqrt{x^2 + y^2}$ is differentiable away from zero, so $r(t)$ is differentiable on s.

It remains to be shown that $\theta(t)$ is differentiable on s. Fix a point t_1 in s. Then t_1 lies in some interval s_i, and we may choose t_0 in s_i with $t_0 < t_1$. Let

$$F(z) = \theta(t_0) - i \int_{\tau(t_0)}^{z} \frac{1}{\zeta}\, d\zeta.$$

Then by Cauchy's integral theorem, $F(z)$ is differentiable at $\tau(t_1)$. Further, $\tau(t) = \sigma_c(t)/|\sigma_c(t)|$ is differentiable on s. Therefore $\theta(t) = F(\tau(t))$ is differentiable at t_1.

In the rest of this document we will use the symbols σ_{R_c} and σ_{P_c} to denote the constructions given above. We will call σ_{R_c} the **rectangular coordinate path** and σ_{P_c} the **polar coordinate path** centered at c and associated with the path σ.

3.6. The Winding Number

We now consider integrals along closed paths in open subsets U of **C**. An important topological property of such a path σ is the number of times σ "winds around" a point not in U. For example:

- A closed path that traverses the unit circle counterclockwise twice winds around the origin with multiplicity 2.
- A closed path that traverses the unit circle clockwise once winds around the origin with multiplicity -1.
- Let σ be a closed path that winds around the origin once counterclockwise, then turns around and winds around the origin once clockwise. σ winds around the origin with multiplicity $1 + (-1) = 0$.

Notice that if a path σ winds around a point not in U, then U is not simply connected: it must have a hole inside σ, so that σ cannot be continuously deformed to a point in U.

We have already seen (§ 3.3) that the integral of $1/z\ dz$ around a closed path $\sigma(t) = e^{it}$ computes $2\pi i$ times the number of times the path winds around the unit circle. We now extend this computation to general closed paths winding around general points.

Definition of the winding number: Fix an open subset U of **C**, a path $\sigma\colon [a, b] \to U$, and a point c in **C** such that σ does not pass through c. We define the following:

$$W(\sigma, c) = \frac{1}{2\pi i} \int_{\sigma} \frac{1}{z - c}\, dz.$$

By § 3.5, we can construct the polar coordinate path σ_{P_c} such that $\sigma = P_c \circ \sigma_{P_c}$, and we can write

$$W(\sigma, c) = \frac{1}{2\pi i} \int_{P_c \circ \sigma_{P_c}} \frac{1}{z - c}\, dz.$$

By equation (11), we have

$$W(\sigma, c) = \frac{1}{2\pi i} \int_a^b \frac{1}{r(t)e^{i\theta(t)} + c - c} (r'(t)e^{i\theta(t)} + ir(t)e^{i\theta(t)}\theta'(t))\ dt$$

$$= \frac{1}{2\pi i} \int_a^b \frac{r'(t)}{r(t)}\ dt + \int_a^b i\theta'(t)\ dt$$

$$= \frac{1}{2\pi i} (\int_a^b d(\ln \circ r) + \int_a^b i\ d\theta)$$

$$= \frac{1}{2\pi i} (\ln(r(b)) - \ln(r(a) + i(\theta(b) - \theta(a))).$$

Since σ is a closed path, $r(b) = r(a)$, so the ln terms cancel. Further, by the way we defined σ_{P_c} in § 3.5, $\theta(b) - \theta(a)$ is 2π times the number of times that σ_{P_c} winds around the origin, which is the same as the number of times that σ winds around c. Thus $W(\sigma, c)$ computes the number of times that σ winds around c. We call $W(\sigma, c)$ the **winding number** of the path σ with respect to the point c.

We extend the winding number to closed chains, by integrating over chains as described in § 3.3:

$$W(\gamma, c) = \frac{1}{2\pi i} \int_\gamma \frac{1}{z - c}\ dz.$$

Then the winding number for a chain is the sum of the weights times the winding numbers for each path in the chain. For example, $W(2\sigma_1 - \sigma_2, c) = 2W(\sigma_1, c) - W(\sigma_2, c)$.

Homologous chains: Fix an open set $U \subseteq \mathbf{C}$ and closed chains γ_1 and γ_2 in U. We say that γ_1 and γ_2 are **homologous** if, for every point c in $\mathbf{C} - U$, we have $W(\gamma_1, c) = W(\gamma_2, c)$. For example:

1. Let U be the complex plane minus the origin, and let c be the origin. Then any two paths that wind once counterclockwise around the origin are homologous.
2. Let U be the complex plane with points c_1 and c_2 deleted. Let $\gamma_1 = \sigma_1 + \sigma_2$, where each σ_i winds once counterclockwise around c_i. Let γ_2 be a single path that winds once counterclockwise around both c_1 and c_2. Then γ_1 and γ_2 are homologous.

We say that a closed chain γ in an open set $U \subseteq \mathbf{C}$ is **homologous to zero** if $W(\gamma, c) = 0$ for every point in $\mathbf{C} - U$. For example, let U be the complex plane with the origin deleted. Then

1. Any closed path that does not wind around the origin is homologous to zero in U.
2. A chain consisting of two paths, one that winds once counterclockwise around the origin and one that winds once clockwise around the origin, is homologous to zero in U.

If two closed paths σ_1 and σ_2 are homotopic on an open set U, then they are homologous. Indeed, for c in the complement of U, $1/(z - c)$ is holomorphic on U; so by Cauchy's integral theorem (§ 3.4), the integrals that compute the winding number for σ_1 and σ_2 are equal.

The homology form of Cauchy's integral theorem: The following is a fundamental result in complex integration theory:

Fix an open set $U \subseteq C$, a closed chain γ in U, and a holomorphic function $f : U \to \mathbf{C}$. If γ is homologous to zero in U, then

$$\int_\gamma f\ dz = 0.$$

For a detailed proof of this theorem, see, e.g., [Lang 1999], IV, § 3.

It follows immediately from the theorem that if closed chains γ_1 and γ_2 are homologous in U, then

$$\int_{\gamma_1} f\ dz = \int_{\gamma_2} f\ dz,$$

because we may apply the theorem to the closed chain $\gamma_1 - \gamma_2$.

Because homotopic pairs of closed paths are homologous, the boxed theorem implies Cauchy's integral theorem as stated in § 3.4, applied to closed paths. The boxed theorem is sometimes called Cauchy's integral theorem as well. Henceforth we will call the statement in § 3.4 the **homotopy form** of Cauchy's integral theorem. We will call the boxed statement the **homology form** of Cauchy's integral theorem.

The homology form of Cauchy's integral theorem tells us that when integrating a holomorphic function f on a closed chain γ in an open set U, the precise shape of γ does not matter: all that matters is how γ winds around points in $\mathbf{C}$ outside of U where f is not holomorphic. We will use this fact in § 6, when we show how to compute path integrals for complex functions expressed as power series.

3.7. Cauchy's Integral Formula

The following theorem relates the value $f(c)$ of a holomorphic function f at a point c to the integral of f along a closed chain that winds around c:

Fix a closed chain γ in U, homologous to zero in U. Let f be holomorphic on U, and let c be a point in U that does not lie on γ. Then

$$\frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{z-c}\, dz = W(\gamma, c) f(c).$$

This statement is called **Cauchy's integral formula**. There is an easy proof if we assume that f has a continuous third derivative in an open ball B centered on c and contained in U. Indeed, by the estimate of the error term in the derivative given in § 9.2 of *The General Derivative*, with $x = c$ and $h = z - c$, we have

$$f(z) = f(c) + Df(c)(z-c) + \int_0^1 D^2 f(c + t(z-c))(z-c)^2\, dt$$

for z in B. Because of the continuity assumption, we may differentiate with respect to z under the integral sign, so we have

$$f(z) = f(c) + (z-c)g(z),$$

where g is holomorphic in B. Let C be a circular path contained in B that winds once around c. Then γ is homologous to $W(\gamma, c)C$ in $U - \{c\}$, so by the homology form of Cauchy's integral theorem (§ 3.6), we have

$$\begin{aligned}
\frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{z-c}\, dz &= \frac{1}{2\pi i} W(\gamma, c) \int_{C} \frac{f(z)}{z-c}\, dz \\
&= \frac{1}{2\pi i} W(\gamma, c) \left(\int_{C} \frac{f(c)}{z-c}\, dz + \int_{C} g(z)\, dz \right) \\
&= \frac{1}{2\pi i} W(\gamma, c)(2\pi i\ f(c) + 0) \\
&= W(\gamma, c) f(c).
\end{aligned}$$

It is a remarkable fact of complex analysis that if a function f is holomorphic at a point c, then it is in fact infinitely differentiable in a neighborhood of c. We shall discuss this issue further in § 4.3 below. So in fact the continuity assumption holds for all holomorphic functions f. The only catch is that some form of Cauchy's integral formula is typically used to prove this property of holomorphic functions. So in practice, we need to prove Cauchy's integral formula in some other way. See, e.g., [Lang 1999], III, § 7 (proof in the special case that γ is a circular path); [Lang 1999], IV, § 2 (proof in the general case).

4. Power Series

A **power series** P over **C** is an infinite sum of terms, each of which is a complex number times a distinct nonnegative power of a variable z. In this section we develop the theory of **formal power series** (i.e., power series considered as algebraic objects) and **complex power series** (i.e., power series considered as complex functions). Then we investigate **analytic functions**, i.e., complex functions $f(z)$ expressible as complex power series. We explore the relationship between analytic functions and holomorphic functions. Using power series, we define the exponential and trigonometric functions over the complex numbers.

4.1. Formal Power Series

Consider a polynomial $p(x)$ in one variable x with coefficients in **C**. We may write such a polynomial as follows:

$$p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n = \sum_{j=0}^{n} a_j x^j.$$

The coefficients a_j are complex numbers. The variable x is a formal variable, not necessarily standing in for any number.

A **formal power series** $P(x)$ over the complex numbers extends this idea by letting j range over all the natural numbers, instead of stopping at some number n:

$$P(x) = a_0 + a_1 x + a_2 x^2 + \cdots = \sum_{j=0}^{\infty} a_j x^j.$$

Notice that we may represent any polynomial as a formal power series, by taking all the coefficients a_j to be zero for $j > n$.

There are at least two reasons to study formal power series:

1. They are interesting algebraic objects in their own right.
2. They have many applications, including the theory of analytic functions in complex analysis.

In this section, we will briefly discuss the algebraic aspects of formal power series. We will take up the theory of analytic functions in the following sections.

The formal power series ring: The set of all formal power series over the complex numbers forms an algebraic structure called a **ring**. A ring is similar to a field in that it supports addition, subtraction, and multiplication. However, unlike a field, not every nonzero element must be invertible with respect to multiplication For example, the integers are a ring; the only invertible integers are 1 and -1 (and for this reason, invertible elements in general rings are called **units**, even though this name really doesn't make sense outside of the integers).

Let us write P_i to denote the coefficient a_i in the formal power series $P(x) = \sum_{j=0}^{\infty} a_j x^j$. Here is how we make the set of formal power series over **C** into a ring:

1. Addition and construction of additive inverses operate term by term. That is, $(P+Q)_j = P_j + Q_j$ and $(-P)_j = -(P_j)$.
2. Multiplication of P by Q operates term by term on the result. Each coefficient $(PQ)_l$ is given by summing all the products $P_j P_k$ such that $j + k = l$:

   $$(PQ)_l = \sum_{j+k=l} P_j Q_k.$$

   This multiplication rule is called the **Cauchy product**. It is well-defined because the sum on the right-hand side is finite.

These rules agree with addition and multiplication of polynomials in the case that all but finitely many of the coefficients of P and Q are zero.

By definition, a formal power series P has a multiplicative inverse if and only if there exists a formal power series P^{-1} such that $P \cdot P^{-1} = 1$ according to the Cauchy product. For example, the formal power series $P(x) = 1 - x$ (with $P_j = 0$ for $j > 1$) has the multiplicative inverse $P^{-1}(x) = G(x)$, where $G(x)$ is the **geometric series**

$$G(x) = 1 + x + x^2 + \cdots.$$

By applying the Cauchy product as described above, you will easily see that $P(x) \cdot P^{-1}(x) = 1$. In general, $P(x)$ has a multiplicative inverse if and only if $P_0 \neq 0$. For the justification of this statement, see § 16 of my paper *Definitions for Commutative Algebra*.

Let P and Q be formal power series, and assume $Q_0 = 0$. By the rule for multiplying power series, we know how to compute Q^j for any $j \geq 0$; for example, $Q^3 = QQQ$. Further, because $Q_0 = 0$, each of the k factors in Q^k contributes at least one power of x to each nonzero term of the result. Therefore $(Q^k)_j = 0$ for $k > j$. With this observation, we define the **composition** of the formal power series P and Q, written $P(Q(x))$ or $(P \circ Q)(x)$, by the following rule:

$$(P \circ Q)_j = (\sum_{k=0}^{\infty} P_k Q^k)_j = \sum_{k=0}^{\infty} P_k (Q^k)_j = \sum_{k=0}^{j} P_k (Q^k)_j.$$

The finite sum on the right is well-defined, because Q^k is defined at each term.

The order of a formal power series: We define the **order** of a nonzero formal power series P to be the smallest natural number j such that $P_j \neq 0$, if such a number exists. We define the order of the zero power series to be ∞. We write the order of P as ord P. For example, ord $G = 0$, where G is the geometric series. From the definition of the Cauchy product, it is clear that for any two formal power series P and Q with finite order we have

$$\text{ord } PQ = \text{ord } P + \text{ord } Q.$$

4.2. Complex Power Series

We wish to use power series to represent complex functions. To do this, we must define the concept of convergence for infinite sums.

Sequences: Let V a vector space. A **sequence** in V assigns one element v_j of V to each natural number $j \geq 0$. We write $\{v_j\}$ to denote a sequence.

Let V be a normed vector space, $s = \{v_j\}$ be a sequence in V, $\varepsilon > 0$ be a real number, and $N \geq 0$ be a natural number. We say that s is **ε-convergent after** N if $|v_j - v| < \varepsilon$ for all $j \geq N$. We say that s **converges** to an element v in V if, for any ε, there exists N such that s is ε-convergent after N. It is straightforward to show that if a sequence s converges to vectors a and b, then $|a - b| = 0$, so $a = b$.

We say that s is **ε-Cauchy after** N if $|v_j - v_k| < \varepsilon$ for all $j, k \geq N$. We say that s is **Cauchy** if, for any ε, there exists N such that s is ε-Cauchy after N. A normed vector space V is **complete** if every Cauchy sequence in V converges in V. The real numbers **R** and the complex numbers **C** are complete, as is any finite-dimensional normed vector space over **R** or **C**.

Series: Fix a vector space V. A **series** of elements of V is an infinite sum

$$S = \sum_{j=0}^{\infty} v_j,$$

where each v_i is an element of V. For each $n \geq 0$, we define the nth **partial sum** to be the finite sum

$$S_n = \sum_{j=0}^{n} v_j.$$

We say that the series S **converges** if the sequence $\{S_j\}$ converges in V. We say that S **converges absolutely** if the series of real numbers $\sum_{j=0}^{\infty} |v_j|$ converges in **R**. If S converges absolutely, then (a) S converges to a vector v; and (b) any reordering of the terms of S converges absolutely and converges to v. The proof of (a) relies on the triangle inequality and is given in [Lang 1999], II, § 2.

Complex power series: A **complex power series** $P(z)$ is a formal power series (§ 4.1) in which we replace the formal variable x by a variable z that stands in for a complex number. A complex power series $P(z)$ thus maps each complex number a to a series

$$P(a) = \sum_{j=0}^{\infty} P_j a^j$$

of complex numbers, which may or may not converge.

Let P be a complex power series and a be a complex number. We say that P **converges at** a if the series $P(a)$ converges absolutely. Equivalently, P converges at a if the series

$$\sum_{j=0}^{\infty} |P_j a^j| = \sum_{j=0}^{\infty} |P_j||a|^j$$

of real numbers converges. By the remarks above, absolute convergence of $P(a)$ implies convergence of $P(a)$.[6] If P does not converge at a, then we say that it **diverges** at a. If a complex power series P converges at all points a in a set $U \subseteq \mathbf{C}$, then we say that P **converges on** U.

The radius of convergence: Fix a complex power series P. We will show that exactly one of the following is true:

1. P converges only at $a = 0$.
2. There exists a real number $r > 0$ such that P converges at a if $|a| < r$ and diverges at a if $|a| > r$.
3. P converges at all complex numbers.

Note that case 2 says nothing about what happens when $|a| = r$. P may converge for some such numbers a and diverge for other such numbers.

Proof: Certainly P converges at $a = 0$. If condition 1 or condition 3 holds, then there is nothing more to prove. So assume neither condition 1 nor condition 3 holds, i.e., there exists a complex number a with $|a| > 0$ such that $P(a)$ converges and a complex number b with $|b| > 0$ such that $P(b)$ diverges. Then the set T of all real numbers $t > 0$ such that $\sum_{j=0}^{\infty} |P_j| t^j$ converges is non-empty and does not contain all real numbers greater than zero. Therefore, by a basic property of the real numbers, T has a least upper bound $r > 0$. Then by definition $P(a)$ diverges if $|a| > r$. Fix a with $|a| < r$, and let S_i be the ith partial sum of the series $S = \sum_{j=0}^{\infty} |P_j||a|^j$. For any indices j and k with $k \geq j \geq 0$, we have

$$|S_k - S_j| = |\sum_{l=j+1}^{k} |P_l||a|^l| = \sum_{l=j+1}^{k} |P_l||a|^l < \sum_{l=j+1}^{\infty} |P_l| r^l. \tag{1}$$

By assumption, the series $\sum_{l=0}^{\infty} |P_l| r^l$ converges, so we can make the right-hand sum as small as desired by taking l large enough. Therefore the sequence $\{S_j\}$ is Cauchy, so it converges, i.e., the series S converges. □

Based on this statement, we define the **radius of convergence** of the complex power series P as follows:

1. In case 1, the radius of convergence is zero.
2. In case 2, the radius of convergence is r.
3. In case 3, the radius of convergence is infinite.

We will say that the radius of convergence is **at least** r if (1) the radius of convergence is a real number $s \geq r$ or (2) the radius of convergence is infinite. A radius of convergence at least $r > 0$ for a power series P defines an open set $B(0, r)$ on which $P(z)$ defines a complex function. In the next section we will extend this idea to convergence on an open ball centered away from zero.

Uniform convergence: Fix a set $U \subseteq \mathbf{C}$, and let V be the vector space of bounded complex functions on U, i.e., functions $f : U \to \mathbf{C}$ such that $|f(z)| \leq r$ for some real number r and all z in U. We may put the **sup norm** on V, i.e., we may assign to each f in V the norm $\|f\|$ equal to the supremum, or least upper bound, of all the real numbers $|f(z)|$ as z ranges over U.

Now let V be the vector space of all complex functions on U, not necessarily bounded. The sup norm is not defined on V. However, for Cauchy and convergent sequences, we just need the norms of the difference vectors $\|f - f_j\|$ and $\|f_j - f_k\|$ to be defined and bounded for large values of j and k. Therefore, we say that a sequence $s = \{f_i\}$ of functions in V is **uniformly Cauchy** if it satisfies the definition of a Cauchy sequence given above with respect to the sup norm. The term "uniform" reflects the fact that for any $\varepsilon > 0$, we may choose one $N \geq 0$ such that all the complex sequences $\{f_j(a)\}$, for all a in U, are ε-Cauchy after N. Similarly, we say that s is **uniformly convergent** if it satisfies the definition of convergence to a function f in V with respect to the sup norm. The term "uniform"

[6] Note, however, that ordinary convergence of the series $P(a)$ does *not* imply that P converges at a; absolute convergence of $P(a)$ is required. This inconsistency in the terminology is unfortunate but standard.

reflects the fact that for any $\varepsilon > 0$, we may choose one $N \geq 0$ such that all the complex sequences sequences $\{f_j(a)\}$, for all a in U, are ε-convergent. If a sequence of functions $\{f_j\}$ is uniformly Cauchy on U, then it converges uniformly to the function $f(z) = \lim_{j \to \infty} f_j(z)$ on U. See [Lang 1999], II, § 2, Theorem 2.1.

Now let P be a power series with radius of convergence at least $r > 0$. Let S be the sequence of partial sum functions $S_j \colon B(0, r) \to \mathbf{C}$ given by $S_j(z) = \sum_{k=0}^{j} P_k z^k$. Equation (1) shows that for each a in $B(0, r)$, the complex sequence $\{S_j(a)\}$ is Cauchy. Further, because the bound on the right-hand side depends only on r, and not on a, (1) shows that the sequence S is uniformly Cauchy on $B(0, r)$. Therefore S converges uniformly to the function $P(z)$ on $B(0, r)$.

The geometric series: From § 4.1, we know that the geometric series $G(x) = 1 + x + x^2 + \cdots$ is the formal inverse of $1 - x$. When r is a real number with $0 \leq r < 1$, $G(r)$ also converges to $1/(1 - r)$. Indeed, let S_n be the nth partial sum

$$S_n = 1 + r + r^2 + \cdots + r^n.$$

By multiplying polynomials we see that for each $n \geq 0$, $S_n \cdot (1 - r) = 1 - r^{n+1}$. Therefore

$$\lim_{n \to \infty} S_n = \lim_{n \to \infty} \frac{1 - r^{n+1}}{1 - r} = \frac{1}{1 - r},$$

since r^{n+1} goes to zero for $0 \leq r < 1$. This convergence for real r shows that $G(z)$ converges absolutely and uniformly on the set of complex numbers z such that $|z| < 1$.

Formal and complex power series: Let P and Q be power series. The formal power series $P(x)$ and $Q(x)$ and the complex power series $P(z)$ and $Q(z)$ are related in the following ways:

1. Assume that P and Q converge on an open set $U \subseteq \mathbf{C}$ containing zero, and $P(a) = Q(a)$ for all points a in U. Then $P(x) = Q(x)$ as formal power series. See [Lang 1999], II, § 3, Theorem 3.2.
2. Assume that P and Q converge on an open ball B centered on zero. Then at all points a of B,
   a. The power series $P + Q$ converges at a, and $(P + Q)(a) = P(a) + Q(a)$.
   b. The complex power series PQ converges at a, and $(PQ)(a) = P(a)Q(a)$.

   See [Lang 1999], II, § 3, Theorem 3.1.
3. Suppose that P converges on an open ball B_1 centered on zero, and $P_0 \neq 0$. Then P^{-1} converges on an open ball B_2 centered on zero. See [Lang 1999], II, § 3, Theorem 3.3. Then by item 2, for all a in $B_1 \cap B_2$, we have $P(a)P^{-1}(a) = (P \cdot P^1)(a) = 1$.
4. Assume that $Q_0 = 0$, and assume the existence of a real number $r > 0$ such that P converges at a whenever $|a| \leq r$. Then for all a such that $\sum_{j=0}^{\infty} |Q_j||a|^j \leq r$, $P \circ Q$ converges, and $(P \circ Q)(a) = P(Q(a))$. See [Lang 1999], II, § 3, Theorem 3.4.

4.3. Analytic Functions

Power series expansions: Fix a complex power series P with radius of convergence at least $r > 0$. Then P defines a complex function $f \colon B(0, r) \to \mathbf{C}$ given by $f(a) = P(a)$ for all a in $B(0, r)$. We can turn this observation around: starting with the function f, we can say that P is a **power series expansion** of f. Because the ball of convergence is centered on zero, we say that P is a **power series expansion at zero**.

We wish to develop power series expansions at other points in $\mathbf{C}$. To do this, we compose a complex power series P with a translation map $T_b = z \mapsto z - b$, where b is a complex number. That is, we define a **power series expansion at** b to be

$$(P \circ T_b)(z) = \sum_{j=0}^{\infty} P_j (z - b)^j.$$

We say that $P \circ T_b$ converges at a if and only if P converges at $a - b$; in this case we have $(P \circ T_b)(a) = P(a - b)$. We say that the radius of convergence of $P \circ T_b$ is the radius of convergence of P. If the radius of convergence of P is infinite, then both P and $P \circ T_b$ converge on the entire complex plane. If the radius of convergence of P is zero, then P converges only at zero, and $P \circ T_b$ converges only at b. If the radius of convergence of P is $r > 0$, then $P \circ T_b$ converges at a if $|a - b| < r$ and diverges at a if $|a - b| > r$. In other words, the set of points where $P \circ T_b$ defines a complex function is $B(b, r)$, the open ball of radius r around b.

Let $f : U \subseteq \mathbf{C} \to \mathbf{C}$ be a complex function, and fix a point b in U. We say that f **has a power series expansion at** b if there exists a real number $r > 0$ and a complex power series P such that for all a in $B(b, r)$, $P \circ T_b$ converges at a, and $(P \circ T_b)(a) = f(a)$. By the uniqueness theorem for complex power series (§ 4.2), P is uniquely determined.

Analytic functions: If a complex function f has a power series expansion at b, then we say that f is **analytic at** b. We say that f is **analytic on** $U \subseteq \mathbf{C}$ if U is open, and f is analytic at every point of U.

If f is analytic at b, then it has a power series expansion $P \circ T_b$ with radius of convergence at least r. Let c be a point in $B(b, r)$. Then we can derive a power series expansion $Q \circ T_c$ for f, where

$$Q_j = \sum_{k=j}^{\infty} P_k \binom{k}{j} c^{k-j}.$$

See [Lang 1999], II, § 4, Theorem 4.1. This formula provides a power series expansion for f at every point c in $B(b, r)$. Therefore f is analytic on $B(b, r)$.

The order of an analytic function: Fix a complex function f with a power series expansion $P \circ T_b$. We define the **order of f at b ,** written $\operatorname{ord}_b f$, to be $\operatorname{ord} P$ (§ 4.1). By the definition of the power series expansion, if f is not identically zero, then $f(b) = 0$ if and only if $\operatorname{ord}_b f = n > 0$. In this case we say that f has a **zero of order** n at b.

Differentiation: Let P be a formal power series. We define the **formal derived series** P' of P according to the rule

$$P'_j = (j + 1)P_{j+1}.$$

This is ordinary term-by-term differentiation, with each term $a_j z^j$ in P yielding the term $j a_j z^{j-1}$ in P'.

Let f be a function analytic at b, whose power series expansion $P \circ T_b$ has radius of convergence at least r. Then

1. The power series expansion $P' \circ T_b$ has radius of convergence at least r.
2. $f(z)$ is holomorphic on $B(b, r)$, with derivative $f'(z) = (P' \circ T_b)(z)$.

See [Lang 1999], II, § 5, Theorem 5.1. This theorem shows that if a function f is analytic at a point b, then it is infinitely differentiable at b.

Integration: Let f be a function analytic at b, whose power series expansion $P \circ T_b$ has radius of convergence at least r, and let σ be a path in $B(b, r)$. Then we may integrate the one form $f\ dz$ term by term on the power series expansion, i.e.,

$$\int_\sigma f\ dz = \sum_{j=0}^{\infty} \int_\sigma P_j z^j\ dz.$$

See [Lang 1999], III, § 2, Theorem 2.4.

Holomorphic and analytic functions: The following result relates holomorphic functions to analytic functions in a very close way:

Fix an open set $U \subseteq \mathbf{C}$, and let $f : U \to \mathbf{C}$ be holomorphic. Then f is analytic on U. Let b be a point in U and $r > 0$ be a real number such that the closed disc D of radius r centered on b is contained in U. Let ∂D be the circle bounding D, oriented counterclockwise. Then f has a power series expansion $P \circ T_b$ given by

$$P_j = \frac{1}{2\pi i} \int_{\partial D} \frac{f(\zeta)}{(\zeta - b)^{j+1}}\ d\zeta,$$

with radius of convergence at least r.

Proof: Let $f_0(z) = f(z + b)$. Then f has the expansion $P \circ T_b$ if and only if $f_0 = f \circ T_b^{-1}$ has the expansion $(P \circ T_b) \circ T_b^{-1} = P$ given by

$$P_j = \frac{1}{2\pi i} \int_{\partial D_0} \frac{f_0(\zeta)}{\zeta^{j+1}}\ d\zeta,$$

where ∂D_0 is the circle of radius r around 0. By Cauchy's integral formula (§ 3.7), putting ζ for z and z for c, we have

$$f_0(z) = \frac{1}{2\pi i} \int\limits_{\partial D_0} \frac{f_0(\zeta)}{\zeta - z}\, d\zeta.$$

By the rules for taking inverses of power series (§ 4.1), we also have

$$\frac{f_0(\zeta)}{\zeta - z} = \frac{f_0(\zeta)}{\zeta}\left(\frac{1}{1 - \frac{z}{\zeta}}\right) = \frac{f_0(\zeta)}{\zeta}\, G(z/\zeta),$$

where $G(z)$ is the geometric series (§ 4.2). Because ζ lies on ∂D_0, we have $|\zeta| = r$. Therefore when $|z| < r$, we have $|z/\zeta| < 1$, so $G(z/\zeta)$ converges absolutely and uniformly, and the formal inverse agrees with the convergent inverse (§ 4.2). Thus we have

$$f_0(z) = \frac{1}{2\pi i} \int\limits_{\partial D_0} f_0(\zeta)\, \frac{G(z/\zeta)}{\zeta}\, d\zeta = \frac{1}{2\pi i} \int\limits_{\partial D_0} f_0(\zeta)\left(\sum_{j=0}^{\infty} \frac{z^j}{\zeta^{j+1}}\right) d\zeta.$$

From the theory of topological spaces we know that the image of a closed, bounded set under a continuous function is closed and bounded. Therefore $f_0(\zeta)$ is bounded on ∂D_0, and we may integrate the infinite sum term by term. Doing this yields

$$f_0(z) = \sum_{j=0}^{\infty}\left(\frac{1}{2\pi i} \int\limits_{\partial D_0} \frac{f_0(\zeta)}{\zeta^{j+1}}\, d\zeta\right) z^j,$$

which is the result we wanted. □

This theorem implies the following facts:

1. If a function f is holomorphic at a point b, then f is infinitely differentiable at b.
2. f is holomorphic at a point b if and only if it is analytic at b.

Note also that P_j is the jth coefficient of the Taylor series expansion at b, so we must have

$$P_j = \frac{f^{(j)}(b)}{j!}$$

for each $j \geq 0$, where $f^{(j)}$ denotes the jth derivative of f. Indeed, we can integrate the Taylor series term by term to compute each coefficient P_j:

$$P_j = \frac{1}{2\pi i} \int\limits_{\partial D} \frac{\displaystyle\sum_{k=0}^{\infty} \frac{f^{(k)}(b)(\zeta - b)^k}{k!}}{(\zeta - b)^{j+1}}\, d\zeta = \sum_{k=0}^{\infty} t_{jk},$$

where

$$t_{jk} = \frac{1}{2\pi i} \int\limits_{\partial D} \frac{f^{(k)}(b)(\zeta - b)^k}{k!(\zeta - b)^{j+1}}\, d\zeta.$$

When $k = j$, we have

$$t_{jj} = \frac{1}{2\pi} \int\limits_{\partial D} \frac{f^{(j)}(b)}{j!(\zeta - b)}\, dz = \frac{1}{2\pi i}\left(2\pi i\, \frac{f^{(j)}(b)}{j!}\right) = \frac{f^{(j)}(b)}{j!}.$$

When $k \neq j$, the one form in the integral is exact on ∂D (§ 3.4). Therefore $t_{jk} = 0$ for $k \neq j$.

Isolated zeros: Let $f : U \to \mathbf{C}$ be a holomorphic function. A **zero** of f is a point a in U such that $f(a) = 0$. We say that a zero a is **isolated** if there exists an open set $V \subseteq U$ containing a such that a is the only zero of f in V.

Using the theory of convergent power series, we can prove that, for any zero a, either $f(z) = 0$ in some open set containing a, or the zero a is isolated. Indeed, we know that in an open set W containing a, $f(z) = P(z - a)$ for some convergent power series $P(z)$. If f is not identically zero on W, then P must have some nonzero terms. Let n be the highest power of $(z-a)$ that appears as a factor in all the terms. Then we have

$$f(z) = (z-a)^n\, Q(z-a),$$

and Q has a nonzero constant term, so $Q(z-a)(a) = Q(0) \neq 0$. Let $H(z) = Q(z-a)$. Then $f(z) = (z-a)^n H(z)$, and H is holomorphic in an open neighborhood of a with $H(a) \neq 0$. By continuity, there exists an open neighborhood $V \subseteq W$ of a such that H is holomorphic in V with $H(b) \neq 0$ for all b in V. But then $f(z) = (z-a)^n H(z)$ is holomorphic with $f(z) \neq 0$ for all $b \neq a$ in V.

Using the topology of **C**, we can then show that if U is open and pathwise connected, then either f is constant on U (i.e., f maps every point in U to the same value) or the zeros of f in U are isolated. See [Lang 1999], III, Theorem 1.2.

4.4. The Exponential and Trigonometric Functions

We now use the theory of power series to define complex versions of the real exponential and trigonometric functions.

The exponential function: Let e^x represent the real exponential function. From the theory of real Taylor series, we obtain the series

$$e^x = \sum_{j=0}^{\infty} \frac{x^j}{j!} = 1 + x + \frac{x^2}{2} + \cdots. \tag{2}$$

See, e.g., *The General Derivative*, § 9.3. By applying the real Taylor formula shown there and using the rule $\dfrac{d}{dx}\, e^x = e^x$, we obtain the series shown. This series converges on all of **R**. It is natural to extend the definition to e^z, defined on all complex numbers z:

$$e^z = \sum_{j=0}^{\infty} \frac{z^j}{j!} = 1 + z + \frac{z^2}{2} + \cdots. \tag{3}$$

Because e^x converges for real $x > 0$, e^z converges absolutely on all of **C**, and uniformly on any open ball $B(0, r)$ with $r > 0$.

By taking the formal derivative (§ 4.3) of the power series (3), we obtain the formula $\dfrac{d}{dz}\, e^z = e^z$. For each $j \geq 0$, let $S_j(z) = \sum_{k=0}^{j} \dfrac{z^k}{k!}$ be the jth partial sum function of e^z. By multiplying polynomials, it is easy to show that $S_j(a+b) = S_j(a) S_j(b)$ for all a, b, and j. Therefore the limits must be the same, so we have

$$e^{a+b} = e^a e^b$$

for all complex numbers a and b.

Let $z = re^{i\theta}$ be a complex number expressed in polar coordinates, with $r > 0$. Then we have $z = e^{\ln r + i\theta}$, where $\ln r$ is the real logarithm of r, and for any complex number a we can define the complex exponentiation function

$$z^a = (e^{\ln r + i\theta})^a = e^{a \ln r + ai\theta}.$$

When a is a real number s, this becomes

$$z^s = e^{s \ln r + si\theta} = r^s e^{i(s\theta)}.$$

That is, taking $re^{i\theta}$ to the real power s takes the magnitude r to the power s and multiplies the angle θ by s.

It is worth reviewing the journey from integer powers, as we first encounter them, through the complex exponential function:

1. We first learn that for a real number a and a nonnegative integer b, a^b means "multiply a by itself b times."
2. Then we learn that $a^{-b} = 1/a^b$ and $a^{bc} = (a^b)^c$.
3. Then we learn that for a real number a and a positive integer b, $a^{1/b}$ means a root, i.e., a number c such that $c^b = a$. c may be irrational, and it may not exist in the real numbers.

4. Using rules 1 through 3, we can take powers a^b, where a is a real number and b is a rational number.
5. By taking limits of sequences of rational powers, we can define powers a^b, where b is an irrational number.
6. In the case that $a = e$, the definition in item 5 agrees with the power series e^x shown in equation (2) applied with $x = b$. We can extend the real power series in equation (2) to complex numbers as shown in equation (3).

Note that we can get through step 5 without power series, but for step 6 we need power series. Also, each of rules 1 through 5 maintains some connection to the original idea of a power as a repeated multiplication. When we pass to the complex exponential e^z, the connection to repeated multiplication is lost. The definition is the complex power series.

The trigonometric functions: Putting z in for x in the real Taylor series, we obtain the series

$$\sin z = \sum_{j=0}^{\infty} \frac{(-1)^j}{(2j+1)!} z^{2j+1} = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots \tag{4}$$

$$\cos z = \sum_{j=0}^{\infty} \frac{(-1)^j}{(2j)!} z^{2j} = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots \tag{5}$$

Again the series converge on all of **C**, because the real series converge for all $r > 0$.

By taking the formal derivative of (4) and (5), we obtain the formulas $\frac{d}{dz} \sin z = \cos z$ and $\frac{d}{dz} \cos z = -\sin z$ By putting iz in for z in (3), multiplying (4) by i, and adding the result to (5) term by term, we obtain the **Euler formula**

$$e^{iz} = \cos z + i \sin z.$$

This formula is valid for general complex numbers z, and in particular when $z = \theta$ is a real number. Thus the formula justifies our use of $e^{i\theta}$ to represent the number $\cos \theta + i \sin \theta$.

The logarithm: In first-year calculus, we learn that for any real number $x > 0$,

1. There is a unique real number $\ln x$ such that $e^{\ln x} = x$.
2. $\ln x = \int_1^x \frac{1}{t}\, dt$. On its domain of definition, $\ln x$ is differentiable, and its derivative is $1/x$.

The function $\ln x$ is called the **logarithm**.

We extend the logarithm to complex numbers as follows. Let $a \neq 0$ be a complex number. We may represent a as $re^{i\theta}$, where $r > 0$ and $0 \leq \theta < 2\pi$. Then

1. There is an infinite set L_a of complex numbers b such that $e^b = a$. L_a is the set of all numbers $\ln r + i(\theta + 2\pi n)$, where n is an integer. Note that $e^{\ln r + i(\theta+2\pi n)} = e^{\ln r} e^{i\theta} e^{2\pi n} = r \cdot e^{i\theta} \cdot 1 = a$.
2. As shown below, on a set $U \subseteq \mathbf{C} - \{0\}$, we may define a function $\log z$ that maps each complex number z in U to an element of L_z, by computing an integral of $1/\zeta\, d\zeta$. On its domain of definition, $\log z$ is holomorphic, and its derivative is $1/z$.

There are many ways to define $\log z$. Here is one. Let U be the complex plane with the origin and the negative real axis deleted. Define $\log z\colon U \to \mathbf{C}$ as follows:

$$\log z = \int_1^z \frac{1}{\zeta}\, d\zeta. \tag{6}$$

Notice the similarity to the definition of $\ln x$ in the real case. U is simply connected, so by § 3.4, the integral in (6) is well-defined, because we may integrate along any path σ from 1 to z and get the same result. By the integral that we computed in § 3.6 for the winding number, for any a in U we have

$$\log a = \log re^{i\theta} = (\ln r - \ln 1) + i(\theta - 0) = \ln r + i\theta,$$

where θ is the oriented angular displacement from the positive real axis to a. For each a in **C**, θ is uniquely defined in the range $-\pi < \theta < \pi$. Thus $\log a$ is a unique member of L_a. By § 3.4, on U we have

$$D \log z = D(z \mapsto \int_1^z \frac{1}{\zeta}\, d\zeta) = \frac{1}{z}\,.$$

More generally, we may do the following:

1. Construct U by deleting any set of points from $\mathbf{C} - \{0\}$ such that U is open and simply connected. For example, we may delete any half-axis of the complex plane, or any ray starting from zero in the complex plane.
2. Pick any point $z_0 = r_0 e^{i\theta_0}$ in U, with $r_0 > 0$ and $0 \le \theta_0 < 2\pi$.
3. Pick any integer n.
4. For z in U, define

$$\log z = \ln r_0 + i(\theta_0 + 2\pi n) + \int_{z_0}^{z} \frac{1}{\zeta}\, d\zeta.$$

Then

$$\log a = \log re^{i\theta} = \ln r_0 + i(\theta_0 + 2\pi n) + (\ln r - \ln r_0) + i(\theta - \theta_0)$$

$$= \ln r + i(\theta + 2\pi n).$$

Again, $\log z$ is differentiable in U with derivative $1/z$.

Note that for any definition of $\log z$, in the case that $z = x$ is a real number, we have $\log x = \ln x$.

5. Isolated Singularities

We now look more closely at the situation where a complex function f is holomorphic everywhere in an open set containing a point p, except that it is undefined at p. This is the situation, for example, with the function $f(z) = 1/z$ in a neighborhood of the point $p = 0$. We call such a point p an **isolated singularity**. It is "isolated" because there exists an open set U containing p such that p is the only singular, or undefined, point in U. Compare the discussion of isolated zeros in § 4.3.

5.1. Laurent Series

Fix a complex function f that has an isolated singularity at b. In general we cannot write a power series expansion $P \circ T_b$ at b, because if we had such an expansion, then we would have $f(b) = P(b - b) = P(0) = P_0$, a complex number. To write an expansion at b, we must extend the notion of power series to include terms with negative powers. We call this extended power series a **Laurent series**.

Definition and properties of Laurent series: We define a Laurent series $L(z)$ as follows:

$$L(z) = \sum_{j=-\infty}^{\infty} L_j z^j,$$

where the coefficients L_j are complex numbers. Given a Laurent series L, we write L^+ to denote the power series

$$L^+(z) = \sum_{j=0}^{\infty} L_j z^j.$$

We write L^- to denote the sum of the negative power terms:

$$L^-(z) = \sum_{j=-1}^{-\infty} L_j z^j.$$

Note that every power series P is a Laurent series L, with $L^- = 0$.

We define the **order** of a Laurent series L, written ord L, to be the smallest integer j such that $L_j \neq 0$ if such an integer exists. Otherwise (1) as before, if L is identically zero, then the order is ∞; and (2) if L has infinitely many negative power terms, then the order is $-\infty$. For example, the order of the Laurent series $1/z$ is -1. Thus a power series P is a Laurent series L with ord $L \ge 0$.

Given two Laurent series L and M of finite order, the Cauchy product LM (§ 4.1) is well-defined; this product together with term-by-term addition makes the set of Laurent series L of finite order, together with $L = 0$, into a ring. Also, for any two Laurent series L and M of finite order, we have the rule

$$\operatorname{ord} LM = \operatorname{ord} L + \operatorname{ord} M.$$

This rule extends the rule that we stated in § 4.1 for the orders of power series.

Let U be a set of complex numbers. We say that a Laurent series L **converges absolutely** on U if both L^+ and L^- converge absolutely on U. Similarly, we say that L **converges uniformly** on U if L^+ and L^- converge uniformly on U. In either case, we regard $L(z)$ as the function given by the sum of the functions $L^-(z)$ and $L^+(z)$.

As with power series, if $f(z) = L(z)$ on a set where L converges absolutely, then we say that f has a **Laurent series expansion at zero**. If $f(z) = (L \circ T_b)(z)$, then we say that f has a **Laurent series expansion at** b. For any complex number a, we say that any convergence property of the Laurent series L at $a - b$ is a convergence property of the expansion $L \circ T_b$ at a.

Computing Laurent series: For any point b in the complex plane and any real numbers r_1 and r_2 with $0 \le r_1 < r_2$, let $A = A(b, r_1, r_2)$ be the set of the points z such that

$$r_1 \le |z - b| \le r_2.$$

When $r_1 = 0$, A is the closed disc D of radius r_2 centered at b. When $r_1 > 0$, A forms a closed annulus, or ring, of points centered at b. It consists of D minus the open ball $B(b, r_1)$.

The following theorem shows how to compute a Laurent series expansion at b:

Fix a complex function f that is holomorphic on an open set U containing an annulus $A = A(b, r_1, r_2)$. Then f has a Laurent series expansion $L \circ T_b$ given by

$$L_j = \frac{1}{2\pi i} \int_{\partial D^j} \frac{f(\zeta)}{(\zeta - b)^{j+1}} \, d\zeta,$$

where $D^j = D^-$ is the disc of radius r_1 centered at b if $j < 0$, and $D^j = D^+$ is the disc of radius r_2 centered at b if $j \ge 0$. $L \circ T_b$ converges absolutely and uniformly for $r_1 < |z| < r_2$.

Note that if U is simply connected, then we may take $r_1 = 0$, and the annulus becomes a disc. In this case $L^- = 0$, and the theorem coincides with the theorem on power series proved in § 4.3. So this theorem extends that one.

Proof: By the same argument that we made in § 4.3, it suffices to show

$$L_j = \frac{1}{2\pi i} \int_{\partial D_0^j} \frac{f_0(\zeta)}{\zeta^{j+1}} \, d\zeta,$$

where $f_0 = f \circ T_b^{-1}$, and D_0^j is the disc D^j with its center translated from b to 0. Let γ be the chain $\partial D_0^+ - \partial D_0^-$. Then γ is homologous to zero in U, because both circles wind once around any point outside U enclosed by A, and zero times around any point outside U not enclosed by A. Therefore for any point z in A, Cauchy's integral formula yields

$$f(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{f_0(\zeta)}{\zeta - z} \, d\zeta = \frac{1}{2\pi i} \int_{\partial D_0^+} \frac{f_0(\zeta)}{\zeta - z} \, d\zeta - \frac{1}{2\pi i} \int_{\partial D_0^-} \frac{f_0(\zeta)}{\zeta - z} \, d\zeta.$$

The argument given in § 4.3 for the power series expansion, applied to the first integral on the right, establishes the result for the terms L_j in L^+.

As to the second integral, we have

$$-\frac{f_0(\zeta)}{\zeta - z} = \frac{f_0(\zeta)}{z} \left(\frac{1}{1 - \dfrac{\zeta}{z}} \right) = \frac{f_0(\zeta)}{z} \, G(\zeta/z),$$

where $G(z)$ is the geometric series. Because ζ lies on D_0^-, we have $|\zeta| = r_1$. Therefore when $|z| > r_1$, we have $|\zeta/z| < 1$, so $G(\zeta/z)$ converges absolutely and uniformly. Thus we have

$$L^-(z) = \frac{1}{2\pi i}\int_{\partial D_0^-} f_0(\zeta)\,\frac{G(\zeta/z)}{z}\,d\zeta = \frac{1}{2\pi i}\int_{\partial D_0^-} f_0(\zeta)\left(\sum_{j=0}^{\infty}\frac{\zeta^j}{z^{j+1}}\right)d\zeta = \frac{1}{2\pi i}\int_{\partial D_0^-} f_0(\zeta)\left(\sum_{j=-1}^{-\infty}\frac{z^j}{\zeta^{j+1}}\right)d\zeta.$$

By the argument we made in in § 4.3, we can integrate term by term, yielding

$$L^-(z) = \sum_{j=-1}^{-\infty}\left(\frac{1}{2\pi i}\int_{\partial D_0^j}\frac{f_0(\zeta)}{\zeta^{j+1}}\,d\zeta\right)z^j.$$

This establishes the result for L^-. □

Uniqueness of Laurent series: We now show that the Laurent series expansion at b is unique:

Fix a complex function f that is holomorphic on an open set U containing an annulus $A = A(b, r_1, r_2)$. If f has Laurent series expansions $L \circ T_b$ and $M \circ T_b$ that converge absolutely and uniformly for $r_1 < \lvert z\rvert < r_2$, then $L = M$.

Proof: By the previous theorem, for each integer j, we have

$$L_j = \frac{1}{2\pi i}\int_{\partial D^j}\frac{f(\zeta)}{(\zeta - b)^{j+1}}\,d\zeta = \frac{1}{2\pi i}\int_{\partial D^j}\frac{\sum_{k=-\infty}^{\infty} M_k\zeta^k}{(\zeta - b)^{j+1}}\,d\zeta.$$

Integrating term by term yields $L_j = \sum_{k=-\infty}^{\infty} t_{jk}$, where

$$t_{jk} = \frac{1}{2\pi i}\int_{\partial D^j}\frac{M_k\zeta^k}{(\zeta - b)^{j+1}}\,d\zeta.$$

By the same argument that we made in § 4.3, $t_{jj} = M_j$, and $t_{jk} = 0$ for $j \neq k$. Therefore $M_j = L_j$ for all integers j. □

5.2. Removable Singularities

Fix a complex function f that is holomorphic on an open set U, except for an isolated singularity at b in U. We say that f has a **removable singularity** at b if there exists a complex number c such that we can extend f to a holomorphic function on all of U by setting $f(b) = c$. Because a holomorphic function is continuous, it is clear that if such a number c exists, then it is unique.

For example, let $U = \mathbf{C}$, and let $f(z) = \dfrac{\sin z}{z}$. Then f has an isolated singularity at zero. By examining the power series expansion of $\sin z$ (§ 4.4), we see that

$$f(z) = \frac{1}{z}\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots\right) = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots.$$

Therefore f has a removable singularity at zero, because we can extend f to a function that is holomorphic on all of $\mathbf{C}$ by setting $f(0) = 1$.

Fix an open set $U \subseteq \mathbf{C}$, a point b in U, and a holomorphic function $f\colon U^* = U - \{b\} \to \mathbf{C}$. If f is bounded on U^*, then f has a removable singularity at b.

Proof: Because U is open, there exists a real number $r > 0$ such that for any $0 < r_2 < r$, f is holomorphic on $D^+ - \{b\}$, where D^+ is the closed disc of radius r_2 centered at b. Then for any $0 < r_1 < r_2$, f is holomorphic on the closed annulus $A = A(b, r_1, r_2)$, and by § 5.1, f has a Laurent series expansion $L \circ T_b$ that converges on A. It suffices to show that $L^- = 0$, because then the power series L^+ will be analytic on all of D^+ and will agree with f on $D^+ - \{b\}$.

We may represent the coefficients of L^- as L_{-j}, for $j \geq 1$. Then for any such j we have

$$L_{-j} = \frac{1}{2\pi i}\int_{\partial D^-} f(\zeta)(\zeta - b)^{j-1}\,d\zeta,$$

where D^- is the closed disc of radius r_1 centered at b. By assumption there exists a bound $B \geq 1$ on $|f(\zeta)|$ for ζ in U^*. Let $0 < \varepsilon < 1$ be a real number such that $\varepsilon < rB$, and let $r_2 = \varepsilon/B$. Then $0 < r_2 < r$, and for ζ on ∂D^- we have $|\zeta - b| \leq \varepsilon/B < 1$, so for any $j \geq 1$ we have

$$|L_{-j}| \leq \frac{1}{2\pi} \int_{\partial D^-} |f(\zeta)||\zeta - b|^{j-1}\ d\zeta \leq \frac{1}{2\pi} \int_{\partial D^-} B\ d\zeta = \frac{1}{2\pi} \cdot \left(2\pi \cdot \frac{\varepsilon}{B}\right) \cdot B = \varepsilon.$$

Since we can do this for any j and any small enough ε, we must have $L_{-j} = 0$ for all $j \geq 1$. □

5.3. Poles

The definition of a pole: Fix a complex function f that has a Laurent series expansion at b. If f has finite negative order at b, i.e., there exists $n > 0$ such that ord $f_b = -n$, then we say that f has a **pole of order** n at b. If $n = 1$, then we say that f has a **simple pole** at b.

For example:

1. $f(z) = 1/z$ has a simple pole at the origin.
2. $f(z) = 1/z^2 + 1/z$ has a pole of order 2 at the origin.

Properties of poles: From the definition of the Laurent series expansion $L \circ T_b$, the following facts are clear:

1. f has a pole of order n at b if and only if the complex function $g(z) = f(z)(z-b)^n$ is holomorphic at n, and g has no zero at b. In this case $f(z) = g(z)/(z-b)^n$ on $U - \{b\}$, where U is an open set containing b, so f is a quotient of holomorphic functions on U.
2. If f is holomorphic on an open set U containing b and $f(b) \neq 0$, then the function $g(z) = f(z)/(z-b)^n$ is holomorphic on $U - \{b\}$ and has a pole of order n at b.

Meromorphic functions: We say that a set $S \subseteq \mathbf{C}$ is **discrete** if, for each point p in S, there exists an open ball B_p centered at p such that no point of S except p lies in B_p. Let f be a complex function defined and holomorphic on an open set U except at a discrete set of points S where f has poles. We say that f is **meromorphic** on U. We say that f is **meromorphic** at a point p if f is meromorphic on an open set U containing p.

5.4. Essential Singularities

Fix a complex function f that has a Laurent series expansion $L \circ T_b$ at b. If $L \circ T_b$ has infinitely many negative terms, i.e., ord $f_b = -\infty$, then we say that f has an **essential singularity** at b. For example, the function $f(z) = e^{1/z}$ has an essential singularity at zero because its Laurent series expansion at zero is

$$L(z) = \sum_{j=0}^{\infty} \frac{1}{z^j\, j!}\,.$$

6. The Calculus of Residues

In this section we apply the theory of integration developed in § 3 to complex functions expressed as Laurent series.

6.1. The Residue Formula

The residue formula for small circles: Fix an open set $U \subseteq \mathbf{C}$, a point b in U, and a complex function f that is holomorphic on $U^* = U - \{b\}$. Let C be a counterclockwise circle of radius $r > 0$ centered on b and contained in U^*. From § 5.1, we know that f has a Laurent series expansion $L \circ T_b$ at b, and that the series converges absolutely and uniformly on U^*. Therefore we may integrate term by term:

$$\int_C f\ dz = \int_C \left(\sum_{j=-\infty}^{\infty} L_j (z-b)^j \right) dz = \sum_{j=-\infty}^{\infty} \left(\int_C L_j (z-b)^j\ dz \right) = \sum_{j=-\infty}^{\infty} L_j I_{j,}$$

where $I_j = \int_C (z-b)^j\ dz$. From § 3.4 we know that $I_j = 2\pi i$ when $j = -1$ and $I_j = 0$ otherwise. Therefore we have the formula

$$\int_C f\ dz = 2\pi i\ L_{-1}.$$

We call L_{-1} the **residue** of f at b, and we write $\operatorname{Res}_b f$ to denote this value. With this notation we obtain the following **residue formula** for integrating around the circle C:

$$\int_C f\ dz = 2\pi i\ \operatorname{Res}_b f. \tag{1}$$

Integration along closed chains: We now show how to reduce integration along a closed chain to a sum of integrals along small circles. Fix an open set $U \subseteq \mathbf{C}$ and a closed chain γ in U that is homologous to zero in U. Let $p_1, \dots, p_n$ be n distinct points in U. For each $1 \le j \le n$,

1. Let D_j be a closed disc centered on p_j, small enough so that D_j and D_k are non-intersecting for all $j \ne k$.
2. Let C_j be the circular boundary of D_j, oriented counterclockwise.

Let

$$\gamma_C = \sum_{j=1}^{n} W(\gamma, p_j) C_j.$$

γ_C is a formal sum of the circles C_j, each counted with multiplicity equal to the number of times that γ winds around the center of C_j.

We wish to show the following:

$$\textit{γ is homologous to γ_C in } U^* = U - \{p_1, \dots, p_n\}. \tag{2}$$

Proof: We must show that $W(\gamma, p) = W(\gamma_C, p)$ for all points p outside U^*. By the definition of γ_C and the linearity of the winding number, for any point p,

$$W(\gamma_C, p) = \sum_{j=1}^{n} W(\gamma, p_j) W(C_j, p). \tag{3}$$

A point p outside U^* is either a point p outside U or a point $p = p_k$ for some $1 \le k \le n$.

Let p be a point outside U. Then

- $W(\gamma, p) = 0$ by hypothesis, because γ is homologous to zero in U.
- For each j, $W(C_j, p) = 0$ because p is outside every disc D_j.

Therefore, by (3), $W(\gamma_C, p) = W(\gamma, p) = 0$.

Now let $p = p_k$, for some $1 \le k \le n$. Because the discs D_j are disjoint, $W(C_k, p) = 1$, and $W(C_j, p) = 0$ for $j \ne k$. Therefore, by (3), $W(\gamma_C, p) = W(\gamma, p_k) = W(\gamma, p)$. □

From (2) and the homology form of Cauchy's integral theorem (§ 3.6), we immediately obtain the following:

For any complex function f that is holomorphic on U^*,

$$\int_\gamma f\ dz = \int_{\gamma_C} f\ dz = \sum_{j=1}^{n} W(\gamma, p_j) \int_{C_j} f\ dz. \tag{4}$$

The residue formula for closed chains: From equations (1) and (4), we obtain the following residue formula for closed chains:

Let $U \subseteq \mathbf{C}$ be an open set, and let γ be a closed chain in U that is homologous to zero in U. Let $p_1, \dots, p_n$ be n distinct points in U, none of which lies on γ. Let $f : U - \{p_1, \dots, p_n\} \to \mathbf{C}$ be holomorphic. Then

$$\int_\gamma f\ dz = 2\pi i \sum_{j=1}^{n} W(\gamma, p_j)\ \operatorname{Res}_{p_j} f.$$

6.2. Computing the Order of a Function

The following result relates the order of a meromorphic function f with the residue of the function f'/f:

> *Let f be a complex function, not identically zero, that is meromorphic at b. Then*
>
> $$\mathrm{Res}_b\ f'/f = \mathrm{ord}_b\ f.$$

Proof: Because f is not identically zero and is meromorphic at b, $\mathrm{ord}_b\ f$ is finite. Let $m = \mathrm{ord}_b\ f$. Then we have the Laurent series expansion

$$f(z) = a_m(z-b)^m + L \circ T_b, \tag{5}$$

where L is a Laurent series of order greater than m. Therefore we have

$$f(z) = a_m(z-b)^m(1 + P \circ T_b), \tag{6}$$

where P is a power series of order greater than zero. Taking the formal derivative of (5), we have

$$f'(z) = ma_m(z-b)^{m-1} + M \circ T_b = ma_m(z-b)^{m-1}(1 + Q \circ T_b), \tag{7}$$

where M is a Laurent series of order greater than $m-1$, and Q is a power series of order greater than zero. Putting (6) together with (7) yields

$$\frac{f'(z)}{f(z)} = \frac{m}{(z-b)}\ \frac{(1 + P \circ T_b)}{(1 + Q \circ T_b)}.$$

From § 4.1 we know that the power series $1+Q$ has an inverse, and this inverse must have constant term 1 in order to yield 1 when multiplying by $1+Q$. Therefore $(1+P)\cdot(1+Q)^{-1} = 1+R$, where R is a power series of order greater than zero, and we have

$$\frac{f'(z)}{f(z)} = \frac{m}{z-b} + \frac{m\ (R \circ T_b)}{z-b} = \frac{m}{z-b} + S \circ T_b,$$

where S is a power series. The result then follows by the definition of $\mathrm{Res}_b\ f'/f$. □

Recall from § 4.3 that a zero of a complex function f is a point b where $f(b) = 0$, and therefore $\mathrm{ord}_b\ f > 0$. The following theorem relates the orders of the zeros and poles of a meromorphic function f to the integral of f'/f along a closed chain:

> *Let γ be a closed chain in an open set $U \subseteq \mathbf{C}$, homologous to zero in U. Let f be meromorphic on U with all of its zeros and poles among the points $p_1, \ldots, p_n$, none of which lies on γ. Then*
>
> $$\int_\gamma \frac{f'}{f}\ dz = 2\pi i \sum_{j=1}^{n} W(\gamma, p_j)\ \mathrm{ord}_{p_j}\ f.$$

Proof: f' is holomorphic away from the poles of f, so f'/f is holomorphic away from the zeros and poles of f. Therefore the residue formula for closed chains applies (§ 6.1). This result then follows from the previous one. □

The following theorem shows how composition with a meromorphic function f transforms a small circle:

> *Let $U \subseteq \mathbf{C}$ be an open set, let b be a point in U, let f be meromorphic on U with no zeros or poles on $U^* = U - \{b\}$, and let C be a circle in U^* centered at b. Then*
>
> $$\mathrm{ord}_b\ f = W(f \circ C, 0).$$

Proof: By the previous theorem,

$$\mathrm{ord}_b\ f = \frac{1}{2\pi i} \int_C \frac{f'}{f}\ dz = \frac{1}{2\pi i} \int_C f^*(\frac{1}{z}\ dz),$$

where f^* represents the pullback with respect to f (§ 3.3). By the argument we made in 3.5,

$$\frac{1}{2\pi i} \int_C f^*(\frac{1}{z}\ dz) = \frac{1}{2\pi i} \int_{f \circ C} \frac{1}{z}\ dz = W(f \circ C, 0).$$

□

For example, let $C: [0, 2\pi] \to \mathbf{C}$ be the circle of radius 1/2 centered at 1 given by $t \mapsto 1 + e^{it}/2$.

1. Let $f(z) = (z-1)^{-1}$. Then f has a simple pole at $z = 1$ and no other zeros or poles. By the theorem, the winding number of $f \circ C$ around zero is -1.
2. Let $f(z) = (z-1)^2 + (z-1)$. Then f has zeros at $z = 0$ and $z = 1$, and no other zeros or poles. By the theorem, the winding number of $f \circ C$ around zero is 2.

6.3. Meromorphic One Forms

Fix an open set $U \subseteq \mathbf{C}$ and a complex function $f: U \to \mathbf{C}$. We say that the complex one form $\omega = f\ dz$ is meromorphic on U if f is meromorphic on U. In this case, we define the residue of ω at a point b in U to be the residue of f at b:

$$\mathrm{Res}_b\ \omega = \mathrm{Res}_b\ f\ dz = \mathrm{Res}_b\ f.$$

The residue of a meromorphic one form is invariant under coordinate transformation, in the following sense. Let $\phi: U \to \mathbf{C}$ be a map defined on the open set $U \subseteq \mathbf{C}$. We say that ϕ is a **local coordinate map** if it is holomorphic on U, and if there exists a holomorphic function $\phi^{-1}: \phi(U) \to U$ that is an inverse function for ϕ, i.e., such that $\phi^{-1} \circ \phi$ is the identity function on U, and $\phi \circ \phi^{-1}$ is the identity function on $\phi(U)$. Because ϕ is the inverse of the continuous function ϕ^{-1}, it takes open sets to open sets.[7] In particular, $\phi(U)$ is an open set.

Fix a local coordinate map ϕ, a chain γ in $\phi(U)$, and a meromorphic one form $\omega = f\ dz$ on $\phi(U)$. Then by the argument given in § 3.5, we can write

$$\int_\gamma \omega = \int_{\phi \circ \phi^{-1} \circ \gamma} \omega = \int_{\phi^{-1} \circ \gamma} \phi^* \omega.$$

Here we compose the map ϕ with the chain by composing ϕ with each path that appears in a term in γ. Thus the one form $\phi^* \omega = f(\phi(z))\phi'(z)\ dz$ on U represents ω under the change of coordinates given by the map ϕ.

Let b be a point in $\phi(U)$, and let C be a small circle in $\phi(U)$ centered at b. Then the winding number of $\phi^{-1} \circ C$ with respect to $\phi^{-1}(b)$ is one, because ϕ^{-1} is one-to-one, so the image of C under ϕ^{-1} does not intersect itself before the endpoint of the path. Therefore

$$\mathrm{Res}_b\ \omega = \frac{1}{2\pi i} \int_C \omega = \frac{1}{2\pi i} \int_{\phi^{-1} \circ C} \phi^* \omega = \mathrm{Res}_{\phi^{-1}(b)}\ \phi^* \omega.$$

This fact is useful in the study of Riemann surfaces, i.e., complex manifolds of complex dimension one. In that application, ϕ is often a composition $\phi = \phi_2 \circ \phi_1^{-1}$, where X is a topological space, U and V are open subsets of X, $\phi_1: U \to \mathbf{C}$ and $\phi_2: V \to \mathbf{C}$ are one-to-one continuous maps giving local coordinates from X to $\mathbf{C}$, and ϕ is holomorphic on the open set $\phi_1(U \cap V)$.

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[7] It is easy to show that the inverse image of an open subset of **C** under a continuous complex function is open. See, e.g., § 3.1 of my paper *Complex Charts on Topological Surfaces*.